RELATIVISTIC VELOCITY ADDITION LAW WITHOUT SECOND POSTULATE OF RELATIVITY

Report

submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

SHWETHA B

(Reg. No. 14401514PH19)

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NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA
SURATHKAL, MANGALORE- 575025

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November, 2015

DECLARATION

I hereby declare that Report of the P.G. Project Work entitled **"Relativistic Velocity Addition Law Without Second Postulate Of Relativity"** which is being submitted to the National Institute Of Technology Karnataka Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the department of Physics, is a *bonafide report of the work carried out by me.* The material contained in this Report has not been submitted to any University or Institution for the award of any degree.

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Place: NITK, Surathkal

Date:

CERTIFICATE

This is to *certify* that the P.G. Project Work Report entitled **"Relativistic Velocity Addition Law Without Second Postulate Of Relativity"** submitted by, Shwetha B, (Register Number: 14401514PH19)as the record of work carried out by her, is *accepted as the P.G. Project Work Report submission* in partial fulfillment of the requirements for the award of degree of Master of Science in the Department of Physics.

Guide
(Name and signature
with Date and Seal)

Chairman-DPGC
(Signature with Date and Seal)

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Thanking You

Shwetha B

ABSTRACT

The aim of this project is to derive the parallel velocity addition law or Einstein's velocity addition law without using the concept of speed of light. Here general formula $w = \frac{(u+v)}{1+\left(\frac{uv}{c^2}\right)}$ is replaced by $w = \frac{(u+v)}{(1+Kuv)}$, where c is replaced by $K^{-\frac{1}{2}}$. It is derived directly from the principle of relativity and a few simple assumptions of smoothness and symmetry, without making use of the principle of constancy of velocity of light. Here value of K can be measured by the careful measurement of speed of any moving object from two inertial frames in relative motion. So here synchronization of clock is not required. With the use of this addition law, one can derive special results such as length contraction, time dilation, longitudinal Doppler effect.

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1 Chapter 1

1.1 Introduction

Till the end of 19th century it was believed that Newton's three laws of motion and idea about the properties of space and time provided a basis on which the motion of matter could be completely understood.The first principle of relativity ever proposed is attributed to Galileo although he did not formulate it precisely. According to that principle, there is no observational consequences of absolute motion. One can only measure one's velocity relative to something else. But theory of electromagnetism formulated by Maxwell disrupted this. Maxwell's theory was extraordinarily successful , but it is inconsistent with certain aspects of the Newtonian ideas of space and time. So a modification of Newton's equation was found to be necessary. It was Albert Einstein who by combining the experimental results and physical arguments of others with his own unique insights first formulated the new principles in terms of which space, time, matter and energy were to be understood. These principles and their consequences constitute the special theory of relativity. Einstein wrote two theories of relativity. In which the 1905 work is known as "special relativity" because it deals only with special case of uniform motion.

According to this principle, every physical law and fundamental physical constant is same for all non-accelerating observers. The motivation for this principle is electromagnetic theory and in fact the field of special relativity was launched by paper entitled *"On the electrodynamics of moving bodies"*. Einstein,s principle is not different from Galileo's. It is the generalization of Galileo's principle.

The principle of relativity is itself a simple idea on which Einstein's theory is based since the time of Galileo. When this theory is applied to very fast moving particle, then its consequence appears very strange, because they are outside our everyday experiences. But results make sense when we think about them carefully.

Modern physics is based on three major theories, that is relativity, quantum mechanics and thermodynamics. The uniqueness about this theory is, it is different from other theories such as electromagnetism because of its generality. Consequently these theories leads to general conclusions which apply to all physical systems and hence they are of fundamental significance.

Special relativity can be deduced from two fundamental postulates:

- Principle of relativity
- Universality of speed of light

Before going to this concept it is better to know some of the very basic definitions which is related to this topic.

Definitions

Events: These are physical phenomenons which occurs independent of any reference frame.

Observers: Observers are those who records the events, both the time and spatial coordinates, in a particular reference frame.

Simultaneous Events: The event which occur when the light signal from two events reach an observer at the same time.

Relativity Of Simultaneity: Two events simultaneous in one inertial frame are not simultaneous in any other frame. This is a consequence of Einstein's postulate.

Proper Time: It is the time difference between two events occurring at the same position. (Denoted by t_0 or τ)

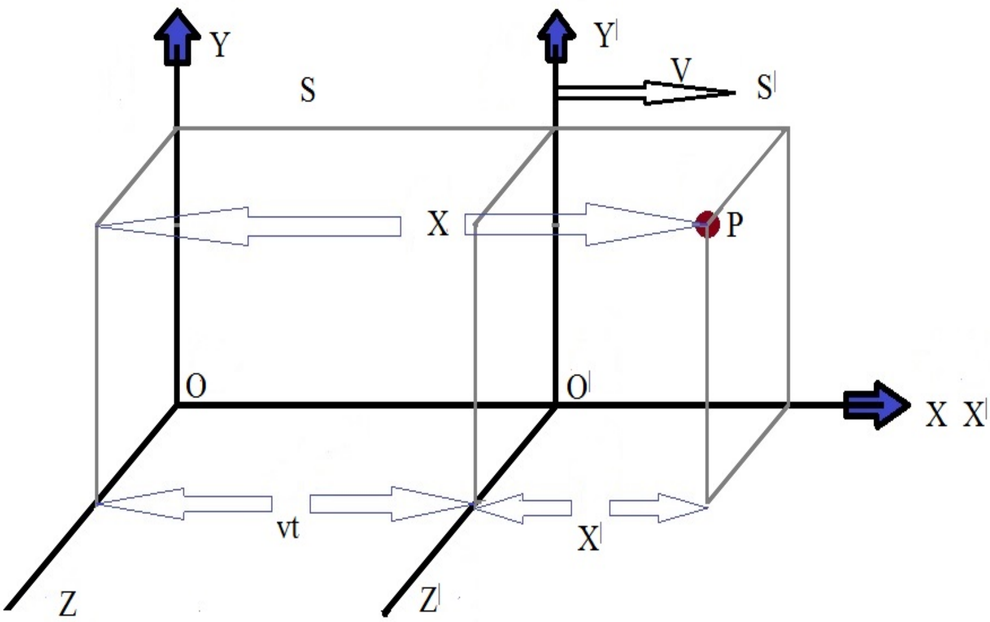
Rest Frame: It is the inertial frame where two events are only separated by time. The frame in which the proper time is measured.

Proper Length: It is the distance between two positions at rest, the length measured in the rest frame.(Denoted by L_0)

1.1.1 Principle Of Relativity And Galilean Transformations

It was first observed by Galileo, in multitude of experiments, mechanical phenomena occur identically in all inertial frames. These observation of Galileo is stated by Newton as "The motions of bodies included in a given space are the same among themselves, when the space is at rest or moves uniformly forward in a straight line". This statement is known as Galilean principle of special relativity, means that all phenomena in nature should be related in same way in all inertial frames. In modern terminology it says that the law of nature must remain invariant as we go from one inertial frame to another.

In order to describe an event we always establish a frame of reference.Frame of reference may be inertial or non-inertial. Consider two inertial frame S and S'. Here S' is moving with the constant velocity v with respect to S.For our convenience we are considering the common x-x' axis.(Shown in figure below) [12]



Let an event occur at P whose space(location) and time of occurrence are measured by each of the observer from S and S' frame and specifies the event by $(x.y, z, t)$ and (x', y', z', t') respectively. Let t and t' is the time recorded by the observers of S and S' respectively. Galilean transformation which relate (x, y, z, t) and (x', y', z', t') are

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

According to classical procedure, length intervals and time intervals are same.That is if meter sticks are of same length when compared at rest with respect to one another, it is implicitly assumed that they are of the same length when compared in relative motion to one another. Similarly if clocks are synchronized at rest, it is assumed that their readings and rates will agree with one another if they are put in relative motion with respect to one another.

For example if P and Q are two events, then time interval between the occurrence of these events is same for each observer. That is,

$$t_P' - t_Q' = t_P - t_Q$$

And also distance or space interval between two points A and B are measured at a given instant is same for each observer. That is,

$$x_B' - x_A' = x_B - x_A$$

Here the endpoints A and B are measured at the same time.

Or we can imagine rod is at rest in primed frame, and therefore moving with velocity v in unprimed one. Then Galilean transformation, which can be written equivalently as,

$$x = x' + vt$$

$$y = y'$$

$$z = z'$$

$$t = t'$$

Here also space interval and time interval measured is same for each of the observers of S and S' frame. That is,

$$t_P - t_Q = t_P' - t_Q'$$

$$x_B - x_A = x_B' - x_A'$$

According to Galilean transformation time interval and space interval measurements made are absolute, that is they are same for all inertial observer independent of relative frame of reference.

1.1.2 Newtonian Relativity

By means of Galilean transformation we can obtain important result of Newtonian mechanics which will give more general form to special relativity

When a particle is in motion, then its position is a function of time. So we can express particle velocity and acceleration in terms of time derivative of position. Then

- Velocity Transformation:

$$u_x{}' = u_x - v$$

$$u_y{}' = u_y$$

$$u_z{}' = u_z$$

- Acceleration Transformation:

$$a_x{}' = a_x$$

$$a_y{}' = a_y$$

$$a_z{}' = a_z$$

From acceleration transformation equation we can say the measured component of acceleration is unaffected by the relative velocity of the reference frames. And this is true as long as v is constant. In classical physics, mass is also unaffected by the motion of the reference frames.Hence the product ma will be same for all inertial observers.If $F = ma$ is taken as definition of force, then F is same for each observer independent of motion of reference frame.That is if $F = ma$ then $F' = ma'$.That implies $F = F'$.

This implies 'Laws of mechanics' are same in all inertial frames. The transformation laws in general will change many quantities but leave some other unchanged. These unchanged quantities re called *invariant* of the transformation.Here we get *acceleration as invariant*. And hence *Newton's laws of motion*.

The general conclusion we have from all this as

Newton's law of motion are identical in all inertial frame of reference

This is the Newtonian principle of relativity and was accepted by all physicists until when Maxwell put together his famous set of equations. [12]

1.1.3 Electromagnetism and Newtonian relativity

Newtonian principle of relativity was successful until the arrival of Maxwell's work in which he given mathematical theory of electromagnetism which provides a successful physical theory of light.
Now it is needed to verify whether laws of electrodynamics is invariant under Galilean transformation or not. If it is invariant then we can say Relativity principle would hold not only for mechanics but all of physics. Now it is needed to verify whether laws of electrodynamics is invariant under Galilean transformation or not. If it is invariant then we can say Relativity principle would hold not only for mechanics but all of physics. Consider a pulse of light traveling to the right with respect to the medium through which it is propagated at a speed c. The medium of light propagation was given the name *Ether*. (In late 19th and early 20th century it was accepted that light or other electromagnetic waves cannot be propagated without a medium). Let that ether frame be S, and the frame which is moving with the uniform velocity v with respect to S is S'. Now speed of light in S frame is exactly $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ = $3 * 10^8 m/s$. And in S' frame that speed varies from $c + v$ to $c - v$ depending on the direction of relative motion. From Maxwell's equation of electromagnetism we deduce the constant $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ = velocity of propagation of plane waves in vacuum. But it is not same for observer in different inertial frame according to Galilean transformation. So in Galilean transformation Newton's laws are same for different inertial observers but not electromagnetic laws.

The fact that Galilean transformation does apply to Newtonian law of mechanics but not to Maxwell's laws of electromagnetism requires us to choose correct consequences among the following possibilities. [12]

1. Relativity principle exist for Mechanics, but not for Electrodynamics. In electrodynamics there is a preferred inertial frame, that is ether frame. Should this

alternative be correct then we would be able to locate the ether frame experimentally.

2. Relativity principle exist for both Mechanics and Electrodynamics. But laws given by Maxwell is not correct.If it is so then we need to reformulate the Maxwell's laws of electromagnetism so that Galilean transformation would apply here also.

3. Relativity principle exist for both Mechanics and Electrodynamics. But laws of mechanics given by Newton is not correct. If it is so then we need to reformulate the mechanical laws so that Galilean transformation would apply here also.

They tested all the three alternatives and confirmed that third possibility was correct. To reject the first and second alternative they conducted some of the experiment which is explained briefly below.

Experimental Basis For Rejecting Alternative 1 and 2

1. **Attempts to locate the absolute frame - The Michelson-Morley Experiment** Experimental setup of Michelson-Morley Experiment is shown in figure below. It is the simplified version of the Michelson interferometer showing how the beam from the source S is split into two beams by the partially silvered mirror M. The beams are reflected by mirrors 1 and 2, returning to the partially silvered mirror. The beams are then transmitted to the telescope T where they interfere, giving rise to d fringe pattern. In this figure v is the velocity of the ether with respect to the interferometer

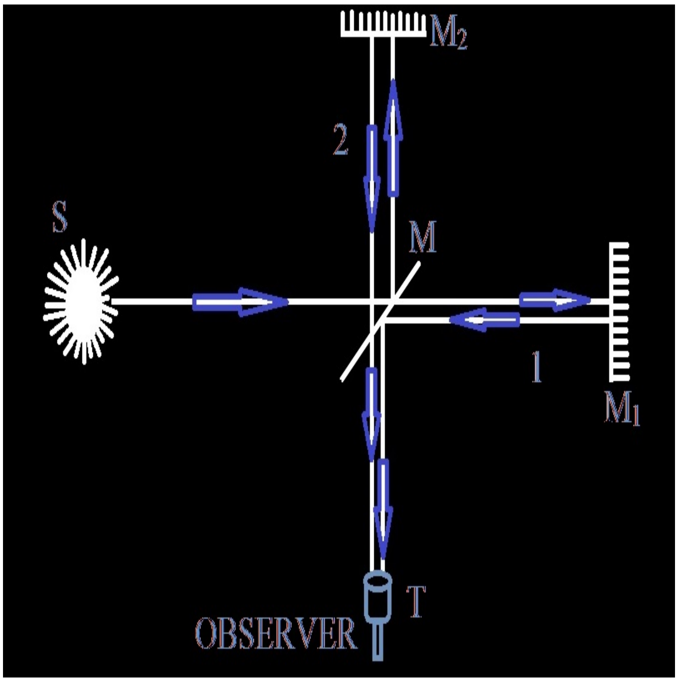


Figure 1: The Michelson interferometer

It is an experiment to measure the motion of the earth through the 'ether', in which light waves are supposed to be occur.Although the experiment is sensitive enough to detect the expected ether drift, none was found.It showed that,

(a) Ether does not exist and so there is no such thing as absolute motion relative to ether.

(b) Speed of light is same for all observers and it is not true that electromagnetic waves need a material medium for its propagation.

So from Michelson-Morley experiment it is clear that there is no preferred inertial frame(Ether frame).So first alternative is not correct.

2. **Attempts to modify Electrodynamics**

From Michelson-Morley experiment we understood that velocity of light has same value in all inertial frames.I If it is so, then velocity of light cannot depend on the velocity of light source relative to the observer. Hence one modification of electromagnetism suggests itself that if we want to avoid the invariance of velocity of light as correct interpretation of Michelson-Morley experiment, we need to assume that velocity of light is connected with the motion of the source rather than with ether.The theories which is based on this assumption are called emission theories.But all emission theories are contradicted by two types of experiment. They are,

(a) de Sitter observation on binary(double) stars.

(b) Michelson-Morley experiment using extraterrestrial source.

From this we conclude that,

- Laws of electrodynamics are correct and do not need any modification.
- Speed of light is same in ll inertial systems, independent of relative motion of source and observer.

From this it is clear that second alternative is also not correct. So Galilean transformation must be replaced and basic law of mechanics given by Newton, which are consistent with those transformations, need to be modified. So some failure of newton's law leads to Special Theory Of Relativity.

1.1.4 The Postulates Of Special Theory Of Relativity

Einstein has inspired to make the postulates of Special Theory of relativity through his study of the properties of Maxwell's equations. In 1905 Einstein published three papers. The first one dealing with photoelectric effect which got him the Nobel prize in 1921. The second one deals with movement of small particles in a fluid(Brownian motion). Third paper was on the electrodynamics of moving bodies. This paper starts with a simple example: That is if a magnet is moved inside a coil, current is generated. If the magnet is kept fixed and the coil is moved, again the same current is produced. This, together with the difficulties in detecting the motion with respect to the author, led Einstein to postulate that *"The same laws of electrodynamics and optics will be valid for all frames of reference for which laws of mechanics holds god."* Which is known as Principle of Relativity.

To understand the implications of Principle of Relativity we need the concept of inertial observer. In terms of inertial observer it is possible to redefine the Principle of Relativity. That is *"All the laws of physics are same for all inertial observers."* Galileo made the similar statement but he referred only to the laws of mechanics. But Einstein generalized this and also he derived unexpected and wonderful consequences from this.

1.1.5 The speed of light and end of Newtonian mechanics

Maxwell"s equation contain a quantity called c, the speed of light, which is given reference to any inertial observer. So if we accept the Principle of Relativity and Maxwell's equations we must conclude that c is same for all inertial observers.

Thus once, Einstein adopted his principle of relativity, he was faced with a choice: either dismiss Newtonian mechanics or dismiss Maxwell's equations. It was impossible for both to be right. Newtonian mechanics was universally accepted in the physics community and its predictions agreed with all experiments(upto 1905). Maxwell's equations, were rather new. were not tested as thoroughly as Newton's and were not universally accepted. Nonetheless Einstein took the daring path of siding with Maxwell and so challenged the Newtonian theory. He was right. Einstein assumed that Newton's mechanics were not a good descriptions of nature under all circumstances. Einstein needed to find a generalization of Newton's mechanics which is consistent with Principle of Relativity and agrees with experiment as well as Newton's theory. He was successful.

Significant conflicting facts between Newton's and Einstein's mechanics become noticeable only at speeds close to c which explains why no problems were detected with Newton's theory before 1905 because all experiments are done at that time at speed very small compared to c.

In conclusion the Principle of Relativity together with Maxwell's equations imply that there is universal speed whose value is same for all inertial observers.

One concept which is modified by Principle of Relativity is that of simultaneity. Everyday experience indicates that the statement "two events happened at the same time(Events are simultaneous)" is universal, and would be verified by any one looking into the matter. one thing that Principle of Relativity does not permit is for some events which occur sequentially, and such that the first affect the second to be inverted in order.

Two postulates of Special Theory of Relativity given by Einstein are

1. **The principle of relativity:** Laws of physics are same in all inertial systems. No preferred inertial system exists.

2. **The principle of constancy of speed of light:**The speed of light in free space has the same value c in all inertial systems.

1.1.6 Lorentz Transformation Equation

Using the postulates of Special Theory of relativity, Galilean transformation equations are replaced by new ones which is derived by Lorentz known as Lorentz Transformation Equations.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Consequences Of Lorentz Transformation Equation

Lorentz transformation equation which is derived from relativity postulates have some consequences for length and time.

1. **First consequence**

   A body's length is measured to be greatest when it is at rest relative to the observer.When it moves with a velocity v relative to the observer its measured length is contracted in the direction of its motion by the factor $\sqrt{1 - \frac{v^2}{c^2}}$, whereas its dimension perpendicular to the direction of motion are unaffected. This is known as *Length Contraction*.

2. **Second Consequence**

   A clock is measured to go at its fastest rate when it is at rest relative to the observer. But when it is moving with the velocity v relative to the observer, its rate is measured to have slow down by a factor $\sqrt{1 - \frac{v^2}{c^2}}$. This is known as *Time Dilation*.

1.1.7 The Relativistic Addition Of Velocities

In classical physics, if we have train moving with velocity v with respect to ground and a passenger on train moves with velocity u' with respect to train, then passengers velocity relative to ground is given by

$$u = u' + v \tag{1.1}$$

In Special Theory of Relativity it is given by

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \tag{1.2}$$

This is *Relativistic* or *Einstein's velocity addition theorem*.

2 Chapter 2

2.1 Literature Survey

- A.R.Lee and T.M.Kalotas (1973) [2] derived Lorentz Transformation by using the principle of relativity alone, without taking the assumption of the existence of universal limiting velocity. In this paper they mentioned that in giving coordinates to events in inertial systems by the method of standard clocks and signals, it is not necessary to use the signal as being light. Any other signal which is believed to travel with uniform speed in the rest frame of the source is also enough.

- N.David Mermin (1983) [10] derived the relativistic addition law for parallel velocities as an immediate consequence of the constancy of the velocity of light, without using Lorentz transformation and without giving attention to the phenomena of Length contraction, time dilation or relativity of simultaneity.

- N.David Mermin (1983) [11] derived the relativistic addition law for parallel velocities directly from the principle of relativity and a few assumptions of smoothness and symmetry and also with the help of gedanken experiments. In this derivation principle of constancy of velocity of light is not used and obtained equation

  $$w = \frac{(u+v)}{(1+Kuv)}$$

  Comparing this with Einstein's equation, K is the replacement of $\frac{1}{c^2}$. Here value of K is determined from careful measurements of speed of any moving object from two inertial frames in relative motion. So here with the use of addition law from the start, one can build up special theory of relativity by avoiding anything that travels with invariant speed.

- Achin Sen (1993) [1] derived kinematic results of special theory of relativity using only first Galilean postulate and by the results of simple thought experiment. In this paper author has derived basic results, including the constancy of universal speed and law of addition of velocities using simple thought experiment. And also it includes discussion of some special results, such as time dilation, Lorentz contraction, Longitudinal Doppler shift etc

2.2 Scopes And Objectives

- Deriving velocity addition law of the form $w = \frac{(u+v)}{(1+Kuv)}$ directly from principle of relativity.

- Determining value of K without making use of principle of constancy of velocity of light.

- From this one can show that relativity does not depend on electromagnetism.

- Deriving Consequences of relativity using velocity addition law.

3 Chapter 3

3.1 Methodology

3.1.1 Introduction

By many ways we can introduce special theory of relativity. The most common method is based on thought experiment. And from this we can derive special results such as time dilation, length contraction and so on. Another method is based on using the invariance of length of space-time four vector and to exploit the resulting constraints to derive transformation formulas. Both of these methods uses two standard postulates, that is principle of relativity and principle of invariance of speed of light. By Galilean postulate we mean: "the laws of physics are same in all inertial frame of reference". Recently there are few attempts to derive Lorentz transformation formulas without using second postulate, that is principle invariance of speed of light. And these approaches are philosophically more satisfying.

The first statement of special theory of relativity was tied to the assertion that the velocity of light in empty space has the value c independent of frame of reference. Reflecting this point, light always had a central role in relativity. It plays an important role in establishing a convention for the synchronization of distant clocks or determining rate of moving clocks or length of moving rigid rods.But relativity is not the branch of electromagnetism. So this subject can be developed without any reference of light. To prove this we should show the parallel velocity addition law of the form

$$w = \frac{u+v}{1+Kuv} \tag{3.1}$$

Where K is universal non negative constant. This equation is compatible with principle of relativity only with natural assumptions of homogeneity, isotropy and smoothness. By comparing (3.1) with actual velocity addition formula, we get

$$K = \frac{1}{c^2}$$

$$c^2 = \frac{1}{K}$$

$$c = \frac{1}{\sqrt{K}}$$

From this we can say $K^{-\frac{1}{2}}$ is invariant velocity.

From this point of view, experiments establishing the constancy of velocity of light are significant only because they determine the value of K. But value of K can be determined from careful measurement of the speed of any moving object from two inertial frames in relative motion. So there is no need of second postulate in the derivation of Lorentz transformation formulas.

So in this derivation only the first postulate and thought experiment are involved. And by looking at two specified events from two reference frames in uniform relative motion, we can derive a set of simple equation which contain all the kinematic theory of special theory of relativity. But for this, two reference frames must be equivalent. And also this method is based only on measurements of length of stationary objects and time interval measured by stationary clocks.

Some assumptions are required here to derive the Lorentz transformation formulas. Here we are using inertial reference frames and within any given inertial frame of reference we are using the concept of distance, time and velocity and last three notions are related only through the relation that the distance covered by the uniformly moving object in a given time is the product of its velocity with time. And we are avoiding the assumption of how distance, velocity and time in one inertial frame are related to those in another , except those constraints placed by the invariance of physical laws. The existence of second postulate is not required as an independent assumption.

This report involves derivation of velocity addition law in two different methods. First method involves two steps. In first step problem of finding the function of two variable specifying the general addition law is reduces that of finding a function of single variable and it is independent of any gedanken(thought) experiment. Second step involves one thought experiment which is designed to determine that unknown function. Second method involves one thought experiment. Both of these method leads to velocity addition formulas in two different ways. Then from these results we can derive special results such as time dilation, length contraction and so on. [1], [11]

3.1.2 Derivation Of Velocity addition Formula

Method 1 [11]

General Form For Velocity Addition Formula

We consider various objects and frames of reference that move in a single common direction. Now consider Two frames A and B and an object which is present in frame A. Let w be its velocity in A, u be its velocity in frame B and v be the velocity of B in A. Then addition Law is a relation

$$w = f(u, v) \tag{3.2}$$

Now introducing the proper frame C and (3.2) can be written in the form

$$V_{CA} = f(V_{CB}, V_{BA}) \tag{3.3}$$

Here f depends only on V_{CB} and V_{BA}. The three velocities are related by (3.3) in the absence of any distinction between two directions of motion. And it must be continue to be related even if all the three signs are changed. So f must be an odd function.

$$f(-x, -y) = -f(x, y) \tag{3.4}$$

It also requires

$$V_{xy} = -V_{yx} \tag{3.5}$$

From (3.4) and (3.5) we can write

$$\begin{aligned} f(V_{CB}, V_{BA}) &= V_{CA} \\ &= -V_{AC} \\ &= -f(V_{AB}, V_{BC}) \\ &= f(-V_{AB}, -V_{BC}) \\ &= f(V_{BA}, V_{CB}) \end{aligned} \tag{3.6}$$

(3.6) implies f must be symmetric in its arguments. So

$$f(x, y) = f(y, x) \tag{3.7}$$

Now by the introduction of another(fourth) frame D, the velocity of D in A can be expressed in two ways.

$$\begin{aligned} f(V_{DB}, V_{BA}) &= V_{DA} \\ &= f(V_{DC}, V_{CA}) \end{aligned} \tag{3.8}$$

Expanding V_{DB} and V_{CA}

$$\begin{aligned} V_{DB} &= f(V_{DC}, V_{CB}), V_{CA} \\ &= f(V_{CB}, V_{BA}) \end{aligned} \tag{3.9}$$

So we get the general solution

$$f(f(x,y),z) = f(x, f(y,z)) \tag{3.10}$$

If f is continuous and differentiable then we can express it as

$$f_2(x,y) = \frac{\partial f(x,y)}{\partial y} \tag{3.11}$$

Now differentiating (3.10) with respect to z gives

$$f_2(f(x,y),z) = f_2(x, f(y,z)) f_2(y,z) \tag{3.12}$$

Setting z=0 in (3.12)

$$f_2(f(x,y),0) = f_2(x,y) f(y,0) \tag{3.13}$$

Here the fact

$$f(y,0) = y \tag{3.14}$$

is used. Now fix x and consider $f(x,y)$ as the function of single variable y, which depends parametrically on x. And also consider $f_2(y,0)$ as a second function of y. Then (3.13) is

$$f_2(f,0) = \frac{df}{dy} f_2(y,0) \tag{3.15}$$

or

$$\frac{dy}{f_2(y,0)} = \frac{df}{f_2(f,0)} \tag{3.16}$$

Now defining a new function $h(z)$ by

$$h(z) = \int \frac{dz}{f_2(z,0)} \tag{3.17}$$

we see that (3.16) requires

$$h(f) = h(y) + constant \tag{3.18}$$

Where constant is independent of y but it depends on parameter x. The symmetry (3.7) requires constant be precisely $h(x)$. We conclude that there must be a function h of single variable such that

$$h(f(x,y)) = h(x) + h(y) \tag{3.19}$$

That implies

$$f(x,y) = h^{-1}[h(x) + h(y)] \tag{3.20}$$

To determine the addition law it is enough to determine the function h. For this, we should know the function $f(x,y)$ at $y = 0$. So (3.17) gives

$$\begin{aligned} h^1(z) &= \frac{1}{f_2(z,0)} \\ &= \frac{1}{(\frac{\partial f(z,y)}{\partial y})}|_{y=0} \end{aligned} \tag{3.21}$$

Equation (3.14) is consistent with (3.19) only if

$$h(0) = 0 \tag{3.22}$$

Equation (3.22) provides boundary condition which is necessary to determine h from (3.21) by integration. Now using some Gedanken experiment determining the form of $h(z)$ to within a single universal constant K.

Precise Form Of Addition Law Using Thought Experiment

A race between tortoise and hare in a long straight train. Race starts from rear of the train towards front. The hare reaches there first and turns back and move(race) towards rear. This hare meets the tortoise again which is still moving front.

Let u is the speed of tortoise and s is the speed of hare in train frame in both the directions. The place where the hare and tortoise meet again is behind the front end by some fraction r of full length of the train. This fraction is frame independent invariant.

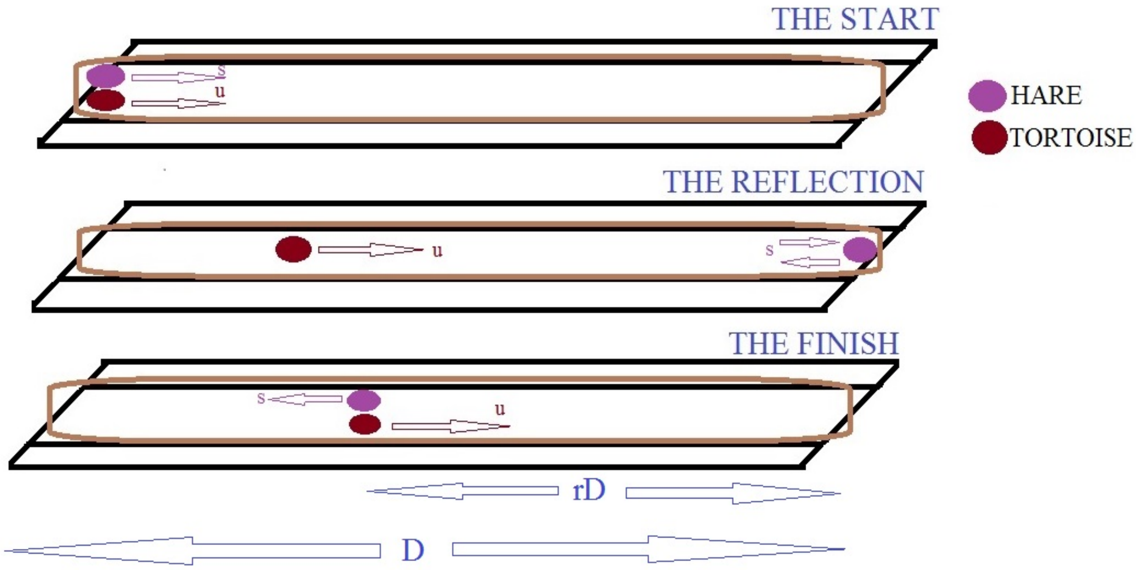


Figure 2: The race between the hare and tortoise as described in the train frame

Now consider r in a frame in which train moving with velocity v. Let w be the velocity of tortoise in v frame and s_1 and s_2 be the speed of hare in the v frame(moving front and back).

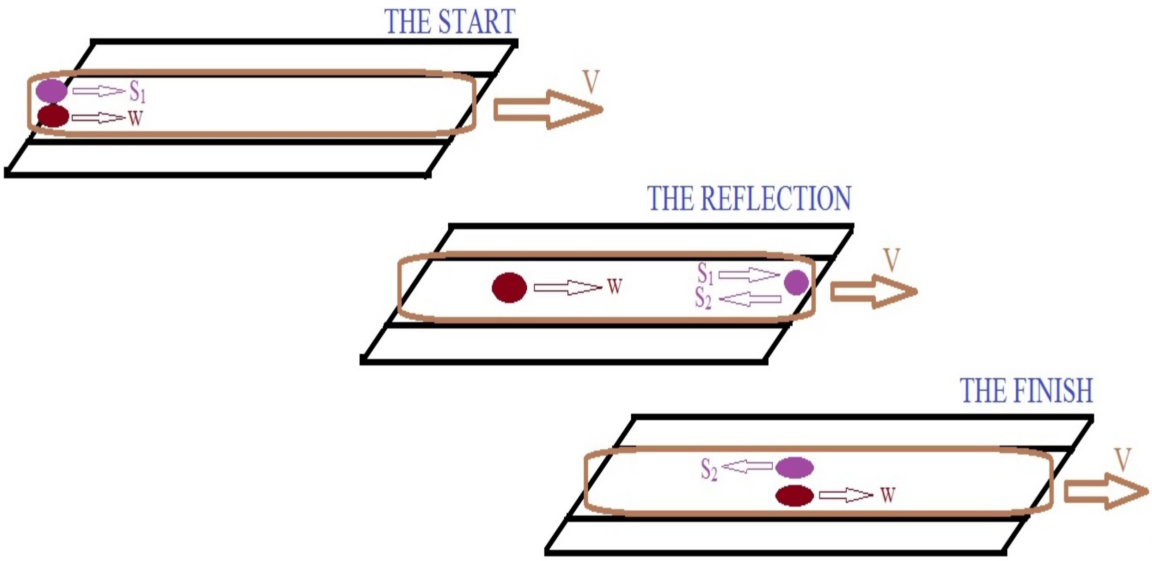


Figure 3: The race between the hare and tortoise as described in the v frame

These speeds are related to train frame speeds by addition law

$$
\begin{aligned}
w &= f(u, v) \\
s_1 &= f(s, v) \\
s_2 &= f(s, -v)
\end{aligned}
\tag{3.23}
$$

Let T is the time taken by hare to move from rear to front. And T' is the time taken by hare to move from front and back to tortoise. And let L be the length of the train. Now to express quantity r entirely in terms of w,s_1,s_2 and v, it is needed to note the following

- Total distance covered by tortoise from the start of the race to its re encounter with hare is $w(T + T')$. This quantity is equal to $s_1T - s_2T'$,
  where s_1T is the distance hare covers in going from rear to front and s_2T is the distance covered by hare from front to the point where it meets tortoise again.

$$w(T + T') = s_1T - s_2T' \tag{3.24}$$

- Distance covered by hare in going from rear to front

$$s_1T = L + vT \tag{3.25}$$

- The distance hare moves in going from front o back to the tortoise is

$$s_2T' = rL - vT' \tag{3.26}$$

Solving (3.25) and (3.26) for T and T' and substituting this in (3.24) and solving for r

$$r = \frac{(s_2 + v)(s_1 - w)}{(s_1 - v)(s_2 + w)} \tag{3.27}$$

This entire calculation is done in v frame. To find the addition law it is needed to determine h which is appearing in (3.20)

$$h^z = \frac{1}{(\frac{\partial f(z,y)}{\partial y})}|_{y=0}$$

$$\frac{\partial f(z, y)}{\partial y}|_{y=0} = \frac{1}{h(z)}$$

Similarly

$$w = f(u, v)$$
$$\frac{\partial f(u,v)}{\partial v}|_{v=0} = \frac{1}{h'(u)}$$
$$s_1 = f(s, v)$$
$$\frac{\partial f(s,v)}{\partial v}|_{v=0} = \frac{1}{h'(s)}$$
$$s_2 = f(s, -v)$$
$$\frac{\partial f(s,-v)}{\partial v}|_{v=0} = \frac{1}{h'(s)}$$

So we can write above equations as

$$\begin{aligned} \frac{\partial w}{\partial v}|_{v=0} &= \frac{1}{h'(u)} \\ \frac{\partial s_1}{\partial v}|_{v=0} &= \frac{1}{h'(s)} \\ \frac{\partial s_2}{\partial v}|_{v=0} &= \frac{-1}{h'(s)} \end{aligned} \tag{3.28}$$

And also as

$$\begin{aligned} v &\to 0 \\ s_1 &\to s \\ s_2 &\to s \\ w &\to u \end{aligned} \tag{3.29}$$

We know r is independent of v and hence $\frac{\partial lnr}{\partial v}$ must vanish

Using (3.28) and (3.29) we can evaluate $\frac{\partial lnr}{\partial v}|_v = 0$

$$r = \frac{(s_2 + v)(s_1 - w)}{(s_1 - v)(s_2 + w)}$$

let $(s_2 + v)(s_1 - w) = f$ and $(s_1 - v)(s_2 + w)$

We can write

$$\frac{\partial lnr}{\partial v} = \frac{1}{r}\frac{\partial r}{\partial v}$$

then

$$\frac{\partial r}{\partial v} = \frac{\partial(\frac{f}{g})}{\partial v}$$

then

$$\frac{\partial(\frac{f}{g})}{\partial v} = \frac{1}{g}\frac{\partial f}{\partial v} - \frac{f}{g^2}\frac{\partial g}{\partial v}$$

By solving we get,

$$\frac{1}{g}\frac{\partial f}{\partial v} = \frac{1}{s(s+u)}\left[(s-u) + \frac{u}{h'(s)} - \frac{s}{h'(u)}\right]$$

and

$$\frac{f}{g^2}\frac{\partial g}{\partial v} = \frac{s(s+u)}{s^2(s+u)^2}\left[u\left(\frac{1}{h'(s)} - 1\right) + s\left(\frac{1}{h^{|}s} - 1\right)\right]$$

Then we get,

$$\frac{\partial lnr}{\partial v} = \frac{2su^2}{(s^2-u^2}\left[\frac{1}{s^2}\left(\frac{1}{h'(s)} - 1\right) - \frac{1}{u^2}\left(\frac{1}{h'(u)} - 1\right)\right] \tag{3.30}$$

We know that r is independent of v. So (3.36) must vanish. So for that

$$\frac{1}{s^2}\left(\frac{1}{h'(s)} - 1\right) = \frac{1}{u^2}\left(\frac{1}{h'(u)} - 1\right) \tag{3.31}$$

In left hand side depends only on s and right hand side depends only on u. So each side must be equal to a constant K which is independent of s or u, and we conclude that

$$\frac{1}{u^2}\left(1 - \frac{1}{h'(u)}\right) = K$$

$$1 - \frac{1}{h'(u)} = Ku^2$$

$$\frac{1}{h'(u)} = 1 - ku^2$$

$$h'(u) = \frac{1}{[1-Ku^2]} \tag{3.32}$$

Using $h(0) = 0$ we can integrate (3.32) to get

$$h(z) = \frac{1}{2\sqrt{K}} ln\left[\frac{1+\sqrt{K}z}{1-\sqrt{K}z}\right] forK > 0$$

$$h(z) = \frac{1}{\sqrt{|K|}}\tan^{-1}(\sqrt{|K|}z) forK < 0 \tag{3.33}$$

When substituting this in $f(x,y) = h^{-1}[h(x) + h(y)]$ that yields

$$w = \frac{(u+v)}{(1+Kuv)} \tag{3.34}$$

Method 2 [1]

Thought experiment

In this method the thought experiment involves three objects, that is fly, platform and train which is moving uniformly with respect to each other. Here train and fly are moving in the same direction. Initially rear end of the train is lined up with rear end of the platform and fly is also in the same position. Finally front end of the train reaches front end of the platform and fly is also in the same point too. Here fly is moving faster than train.

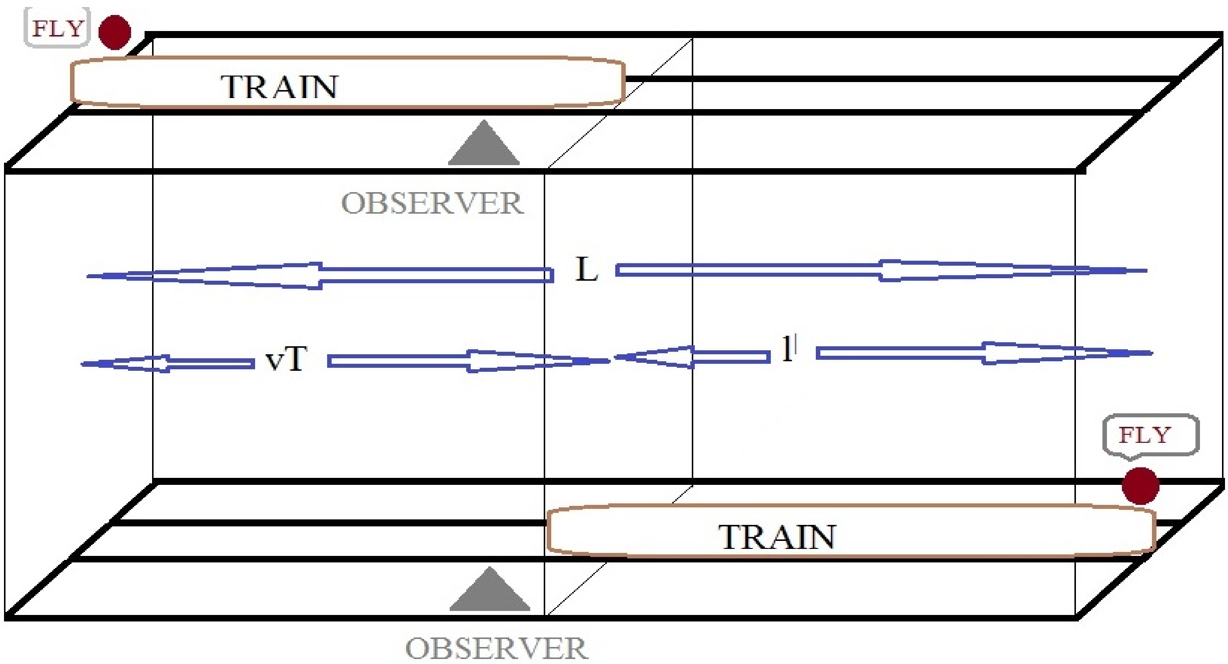


Figure 4: events according to the observer in platform frame (S)

Now we have two observers. Observer S on platform, and another observer s, riding the train. Now using first first postulate we should derive the relationship involving time and space intervals between thees two events as measured by these two observers.

According to observer S, Train is moving with uniform speed v and length of the platform is L. And T is the time elapsed between two events. According to observer s, Platform moves backward with same uniform speed v and lengthy of the train is l. And t is the time elapsed between two events.

According to S, length of the train is l' is the length of the platform minus distance covered by the train

$$l' = L - vT$$

And also according to S, time taken by the fly to reach the front end from rear end is T. So according to S, speed of fly is

$$w = \frac{L}{T}$$

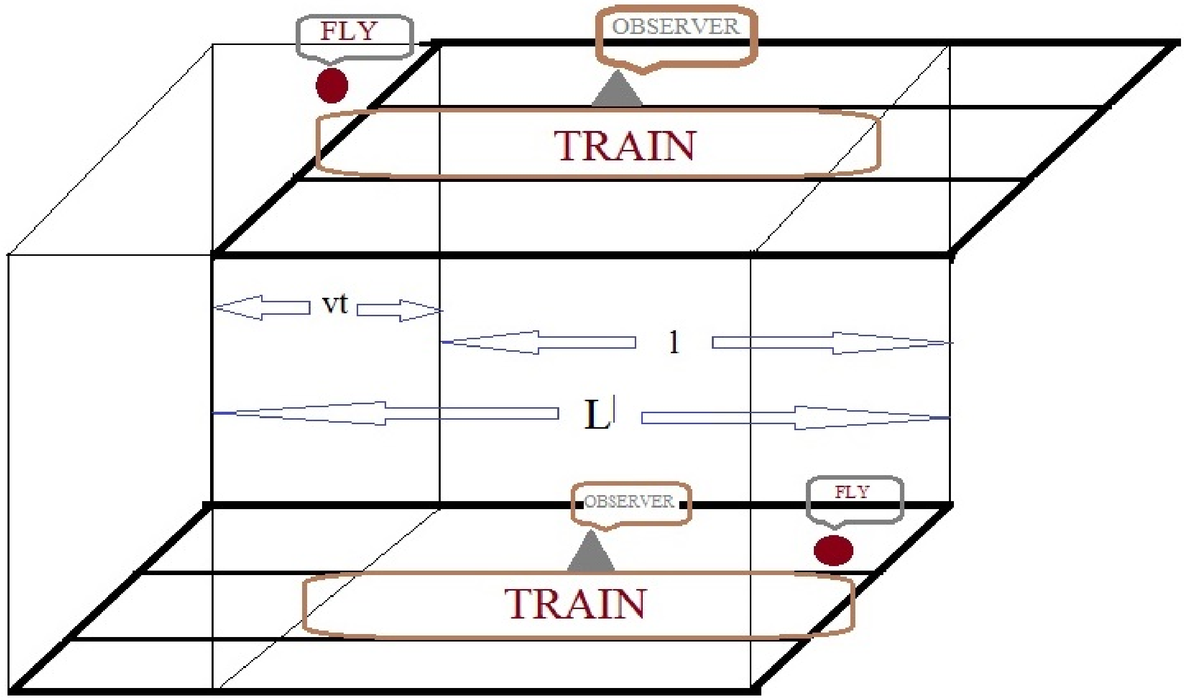


Figure 5: events according to the observer in train frame (s)

According to s. length of the platform is L' is the length of the train plus distance covered by platform

$$L' = l + vt$$

and also according to s, the time taken by the fly to reach the front end from rear end is t. So according to s, speed of fly is

$$u = \frac{l}{t}$$

observer	S(Platform)	s(Train)
Time Elapsed	T	t
Length Of Platform	L	$L' = l + vt$
Length Of Train	$l' = L - vT$	l
Speed Of Fly	$w = \frac{L}{T}$	$u = \frac{l}{t}$

The ratio of the length of the given object calculated by the moving observer to that measured by the observer at rest with the object must be equal to the same scale factor f, which can only depend on the relative velocity v between observers. If two ratios $\frac{l'}{l}$ and $\frac{L'}{L}$ were different, one of the two inertial frames would measure a consistently larger set of lengths, and hence it would be possible to detect an effect of possible motion. so we nust have,

$$\frac{l'}{l} = \frac{L'}{L} = f \tag{3.35}$$

From (3.35) we can write

$$\frac{L - vT}{l} = \frac{l + vt}{L} = f \tag{3.36}$$

So we get

$$\frac{L - vT}{l} = f$$

That implies $L - vT = lf$

$$\frac{l + vt}{L} = f$$

That implies $l + vt = Lf$

Now we have,

$$\begin{aligned} vT &= L - lf \\ vt &= Lf - l \end{aligned} \tag{3.37}$$

So we get

$$\frac{t}{T} = \frac{Lf - l}{L - lf}$$

$$\frac{T}{t} = \frac{L - lf}{Lf - l} \tag{3.38}$$

Now we can derive velocity addition formula, that is expression for w in terms of u and v.

According to S,

$$\begin{aligned} w &= \frac{L}{T} \\ &= \frac{L}{\left[\frac{t(L-lf)}{Lf-l}\right]} \\ &= \frac{L(fL-l)}{t(L-lf)} \end{aligned} \tag{3.39}$$

According to s,

$$\begin{aligned} u &= \frac{l}{t} \\ &= \frac{l}{\left[\frac{T(fL-l)}{L-fl}\right]} \\ &= \frac{l(L-fl)}{T(fL-l)} \end{aligned} \tag{3.40}$$

From (3.37) we can write v as

$$v = \frac{fL-l}{t} = \frac{L-lf}{T}$$

Now adding the expressions of u and v, we get

$$\begin{aligned} (u+v) &= \left(\frac{l(L-fl)}{T(fL-l)}\right) = \left(\frac{L-lf}{T}\right) \\ &= \frac{(L-lf)}{T}\left[\left(\frac{l}{fL-l}\right)+1\right] \\ &= \left(\frac{L-fl}{T}\right)\left(\frac{fL}{fL-l}\right) \\ &= \left(\frac{fL}{T}\right)\left(\frac{L-fl}{fL-l}\right) \\ &= \left(\frac{fL}{T}\right)\left(\frac{T}{t}\right) \\ &= \frac{fL}{t} \end{aligned} \tag{3.41}$$

And

$$\frac{u}{v} = \frac{l}{fL-l} \tag{3.42}$$

Using (3.40),(3.41) and (3.42), we can derive expression for w

$$w = \frac{(u+v)}{\left[1+(1-f^2)\left(\frac{u}{v}\right)\right]} \tag{3.43}$$

From (3.43) remembering that f is a function of v alone, we must therefore have

$$\frac{(1-f^2)}{v} = Kv$$

$$1 - f^2 = Kv^2$$

Now substituting above equation in (3.43), we get

$$w = \frac{u+v}{1+Kuv} \tag{3.44}$$

Where K is universal non negative constant having dimension of inverse square velocity. Defining $K = \frac{1}{c^2}$, where c is universal speed of nature, we get

$$1 - f^2 = \frac{v^2}{c^2}$$

$$f^2 = 1 - \frac{v^2}{c^2}$$

$$f = \left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}} \tag{3.45}$$

(3.45) gives expression for scale factor f as a function of v, Now we can write (3.44) as

$$w = \frac{(u+v)}{\left[1 + \left(1 - 1 + \frac{v^2}{c^2}\right)\left(\frac{u}{v}\right)\right]}$$

$$w = \frac{u+v}{1+\frac{uv}{c^2}} \tag{3.46}$$

(3.46) indicates that as u tends to c, w also approaches to c, for any value of v, That is

$$w = \frac{c+v}{1+\frac{cv}{c^2}}$$

$$w = \frac{(c+v)c^2}{c^2+vc}$$

$$w = c$$

Thus we find that anything which is moving in universal speed c would have the same relative velocity c in any inertial frame of reference. The speed c,therefore, is also the speed limit of nature.

3.1.3 Special results

we have (3.38)

$$\frac{T}{t} = \frac{L - lf}{fL - l}$$

$$t(L - lf) = T(fL - l) \tag{3.47}$$

This equation is symmetric in space and time co-ordinates measured in two reference frames. That is under the exchange of t to T and l to L. As noted in connection with our thought experiment, the pairs (L, T) and (l, t) stand for space and time intervals between same two events measured in the reference frames S and s respectively.

Time Dilation

Consider a stationary clock in S frame measuring a proper time interval τ, we have $L = 0$ and $T = t$, Then (3.46) becomes,

$$t = \frac{\tau}{f}$$

$$t = \frac{\tau}{(1 - \frac{v^2}{c^2})^{\frac{1}{2}}} \tag{3.48}$$

Thus a moving clock appears to be run slower.

Lorentz contraction

Consider a stationary rod in S frame of proper length(measured in S) Λ, then we have $L = \Lambda$. (3.47) implies that from s frame of reference, a simultaneous measurement of distance between end points of rod is given by

$$t(L - fl) = T(fL - l)$$

$t = 0$ and $L = \Lambda$ gives

$$T(f\Lambda - l) = 0$$

$$f\Lambda = l$$

$$l = \Lambda \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{3.49}$$

This implies moving rod appears to be shorter.

Longitudinal Doppler shift

Let us consider monochromatic electromagnetic radiation from a source which is stationary in S frame. In this frame, the space and time intervals between two successive wave crest are $L = \lambda$ and $T = \frac{\lambda}{c}$ respectively. Viewed from source s, speed of waves will still be c, but wavelengtgh will be $l = \lambda^{|}$ and $t = \frac{\lambda^{|}}{c}$. Putting these values in (3.47), we get

$$t(L - fl) = T(fL - l)$$

$$\frac{\lambda'}{c}(\lambda - f\lambda') = \frac{\lambda}{c}(f\lambda - \lambda')$$

$$f(\lambda')^2 + f\lambda^2 - \lambda\lambda' - -\lambda\lambda' = 0$$

$$f(\lambda')^2 + f\lambda^2 - 2\lambda\lambda' = 0$$

Now dividing above equation by λ^2, we get

$$f\left(\frac{\lambda'}{\lambda}\right)^2 - 2\left(\frac{\lambda'}{\lambda}\right) + f = 0$$

This is quadratic in $\frac{\lambda^{|}}{\lambda}$. By solving this equation we get

$$\frac{\lambda'}{\lambda} = \frac{2 \pm \sqrt{4 - 4f^2}}{2f}$$

$$\frac{\lambda'}{\lambda} = \frac{2 \pm \sqrt{4(1 - f^2)}}{2f}$$

$$\frac{\lambda'}{\lambda} = \frac{2 \pm 2\sqrt{1 - f^2}}{2f}$$

$$\frac{\lambda'}{\lambda} = \frac{1 \pm \sqrt{1 - f^2}}{f}$$

We know $f^2 = 1 - \frac{v^2}{c^2}$. Substituting this in above equation, we get

$$\frac{\lambda'}{\lambda} = \frac{1 \pm \sqrt{1 - (1 - \frac{v^2}{c^2})}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\frac{\lambda'}{\lambda} = \frac{1 \pm \frac{v}{c}}{\sqrt{(1 + \frac{v}{c})(1 - \frac{v}{c})}}$$

$$\frac{\lambda'}{\lambda} = \frac{(1+\frac{v}{c})^{\frac{1}{2}}}{(1-\frac{v}{c})^{\frac{1}{2}}}$$

$$\frac{\lambda^{|}}{\lambda} = \frac{(1-\frac{v}{c})^{\frac{1}{2}}}{(1+\frac{v}{c})^{\frac{1}{2}}}$$

Thus we arrive at Longitudinal Doppler shift results for approaching and receding sources.

Lorentz Transformation equations

We have (3.36)

$$\frac{L - vT}{l} = \frac{l + vt}{L} = f$$

From thees equations we get

$$L = \frac{l + vt}{f}$$

$$T = \frac{L - lf}{v}$$

and also we have $f^2 = 1 - \frac{v^2}{c^2}$. Now substituting this in above equation, we get

$$L = \frac{l + vt}{1 - \frac{v^2}{c^2}}$$

$$T = \frac{t + \frac{vl}{c^2}}{1 - \frac{v^2}{c^2}}$$

These are Lorentz transformation formulas.

Proper Time, Proper Length And Invariant Interval Between Two events

Let us look at two events from fly's frame of reference. To begin with fly finds left end of platform and rear end of train to coincide with its own position. For fly, both platform and train found to receding backwards with speed w and u respectively. After some time τ measured in fly's frame of reference , the fly finds right end of the platform and front end of the train alongside itself. From this fly concludes that platform and train take same time to move through their respective lengths. since platform is moving with speed w relative to fly, the length of platform according to fly's measurement will be equal to rest length L multiplied by scale factor $f(w)$. Similarly length of train is $f(u)$ multiplied by l. Thus we have

$$\tau = \frac{f(u)l}{u} = \frac{f(w)L}{w}$$

where

$$f(u) = \sqrt{1 - \frac{u^2}{c^2}} \tag{3.50}$$

$$f(w) = \sqrt{1 - \frac{w^2}{c^2}} \tag{3.51}$$

Squaring (3.51) and (3.52) and using $l = ut$ and $L = wT$

$$f(u) = \sqrt{1 - \frac{u^2}{c^2}}$$

$$\frac{\tau u}{l} = \sqrt{1 - \frac{u^2}{c^2}}$$

$$\frac{\tau^2 u^2}{l^2} = 1 - \frac{u^2}{c^2}$$

$$\tau^2 = t^2 - \frac{l^2}{c^2}$$

Similarly we will get expression for τ^2 in terms of L as

$$\tau^2 = T^2 - \frac{L^2}{c^2}$$

That is

$$\begin{aligned} \tau^2 &= t^2 - \frac{l^2}{c^2} \\ \tau^2 &= T^2 - \frac{L^2}{c^2} \end{aligned} \tag{3.52}$$

From the above expression following expressions can be made

- The quantity τ is known as proper time between the events because this time is measured in fly's frame of reference in which two events happen to be local. Here τ is shortest time interval between two events as measured by any inertial frame of reference. The interval between such a pair of events is time like. (3.53) also tells that it is impossible to find the frame of reference in which these two events will be simultaneous.

- It quantity τ is imaginary then the separation between two events is known as space-like. Clearly it is impossible to send a physical signal from one event to another separated by a space-like interval. On the other hand. it is impossible to find a reference frame in which two events appeared to be simultaneous. and in this frame , space separation between the events is called the proper length.
  From (3.53) we can write

$$\begin{aligned} \Lambda^2 &= l^2 - c^2t^2 \\ \Lambda^2 &= L^2 - c^2T^2 \end{aligned} \tag{3.53}$$

- (3.53) and (3.54) together can be used to define general invariant separation between a pair of events. In the special case where Λ^2(and hence τ^2 is zero, then the separation between the event is time like.

It is clear from the above discussion that nature of interval is also a frame invariant concept.

4 Chapter 4

4.1 Conclusion

Einsteins velocity addition law of the form $w = \frac{u+v}{1+Kuv}$ can be derived without using Lorentz transformation equation. Actual derivation of Lorentz transformation equation requires synchronization of clock which depends on speed of light. And from that lorentz transformation equation we can derive velocity addition law. So that implies derivation of velocity addition law depends on the concept of seed of light. But here synchronization of clock is not used for the derivation of velocity addition law. and from this velocity addition law we can derive special results such as time dilation, lenngth contraction etc. That implies special relativity can be derived without using the concept of speed of light.

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