NEUTRINO OSCILLATIONS AND A SPIN ONE HALF FERMIONIC FIELD WITH MASS DIMENSION ONE

PROJECT REPORT

submitted in partial fulfillment of the requirement for the degree of

MASTER OF SCIENCE

in

PHYSICS

by

DIVYAA SHREE R

Roll No. : 176PH012

DEPARTMENT OF PHYSICS

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA(NITK),

SURATHKAL, MANGALORE.

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DECLARATION

I hereby declare that the report of the postgraduate project work entitled "NEUTRINO OSCILLATIONS AND A SPIN ONE HALF FERMIONIC FIELD WITH MASS DIMENSION ONE" which is submitted to National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the degree of Master of Science in the Department of Physics, is a bonafide report carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree. In keeping with the general practice in reporting scientific observations due acknowledgement has been made whenever the work described is based on the findings of other investigators.

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CERTIFICATE

This is to certify that the project entitled "**NEUTRINO OSCILLATIONS AND A SPIN ONE HALF FERMIONIC FIELD WITH MASS DIMENSION ONE**" is an authenticated record of work carried out by **DIVYAA SHREE R**, Reg. No. : 176PH012 in partial fulfillment of the requirement for the award of the degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology Karnataka, Surathkal, during the period 2018-2019.

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Chairman - DPGC

(Signature with date and seal)

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ABSTRACT

According to the Standard Model, which is considered to be the fundamental model of Particle physics, the leptonic neutrinos are considered to be massless since the time they were discovered. But the question of missing neutrinos in contrast to their abundant nature, gave rise to the concept of *neutrino oscillations.* The conversion of neutrinos from one flavor to another in mid-way of their travel causes their oscillations. The oscillation probabilities of neutrinos in vacuum and in matter are discussed in detail. Further, the flavor-conversion of neutrinos imply that they must have a particular mass whose lower limit is yet to be determined. The see- saw mechanism which can account for the extremely low masses of neutrinos has also been studied.

In 2004, an unexpected theoretical discovery of a spin one half matter field with mass dimension one was reported. This quantum field is based on a complete set of eigenspinors of the charge conjugation operator. It exhibits unusual properties under charge conjugation and parity operation. Due to this reason, they belong to a Non-Standard Wigner Class(NSWC). As a result, the theory also exhibits non-locality with $(\mathcal{CPT})^2 = -\mathbb{I}$. Its dominant interaction with known forms of matter is through Higgs and gravity. This allows us to contemplate it as a first principle candidate for dark matter.

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Part I

Neutrino Oscillations

Chapter 1

Introduction

In 2015, the Nobel Prize in Physics was awarded to Takaaki Kajita and Arthur B. McDonald for the "Discovery of Neutrino Oscillations" by their respective teams. This discovery gave us a new concept of flavour conversion of neutrinos and hence broke the myth that neutrinos are massless. It has presented a compelling experimental evidence for the incompleteness of the Standard Model of particle physics and has forced us to look into the related problems like the nature of the neutrino, mass of the neutrinos and the possible charge-parity violation among leptons.[Ber16]

1.1 Early History of Neutrinos

It all started in 1914, when James Chadwick proposed that the β-spectrum from the radioactive decay of an element was continuous i.e. the electron emitted from the same element had different values of energy. This demonstration created a stir as it violated the fundamental principle of conservation of energy. In 1930, W.Pauli proposed a hypothetical, weakly interacting, spin- $\frac{1}{2}$ fermion with a mass similar to that of proton as a solution to the puzzle and named it *neutron*. However, E.Fermi thought that a much more massive particle discovered by Chadwick could possibly bear that name and called Pauli's elusive particle as a *neutrino*[Raj16]

The neutrinos were experimentally detected by F.Reines and C.L.Cowan Jr. through inverse β-decay in a nuclear reactor. The Cowan - Reines reactions inspired from Pontecorvo's idea is given by,

$$\bar{\nu_e} + p \rightarrow n + e^+ \tag{1.1}$$

Further, detection of solar neutrinos was very important to verify the energy production process in the Sun. About fifty years ago, Ray Davis started his pioneering experiments and could finally detect only one-third of the neutrinos reaching the Earth from the Sun. The contemplation about the missing neutrinos came to be known as the *Solar neutrino puzzle*. In 1998, Japan's Super Kamioka experiment discovered that neutrinos have masses and

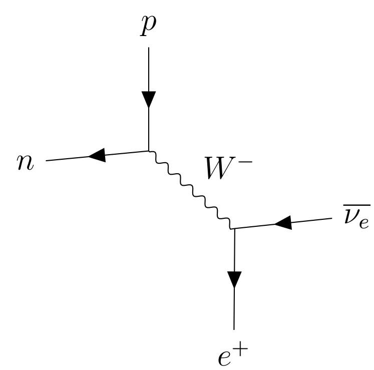


Figure 1.1: Inverse β - decay

gave a solution to the *Atmospheric neutrino problem*, which arised due to lesser detection of cosmic ray- produced neutrinos than expected. Finally, the puzzle of missing solar neutrinos was also resolved by the Sudbury Neutrino Observatory in 2002. The reason behind these missing ghost particles was their oscillations.[Raj16]

1.2 Scope and Objective

1.2.1 Scope

Neutrinos are important because the existence of their masses guarantee us of some Physics beyond our Standard Model. There are many unsolved questions about neutrinos. Scientists still are in the look out for new flavour of these particles and in studying their mass hierarchy. There is a whole new class of neutrinos called the sterile neutrinos(neutrinos with right-handed chirality) which is to be detected. These sterile neutrinos are expected to explain some unxplained phenomena such as the dark matter and baryogenesis.

1.2.2 Objective

- To study the origin of the phenomenon of neutrino oscillations
- To study and analyse the phenomenon of neutrino oscillations in vacuum
- To study the phenomenon of neutrino oscillations in matter
- To understand the see-saw mechanism to account for the extremely low masses of neutrinos.

Chapter 2

Neutrino oscillations in vaccuum

2.1 Introduction

According to the Standard Model, neutrinos are massless and chargeless particles. They undergo only weak interactions. Neutrinos come in three flavours: electron neutrinos (ν_e), muon neutrinos (ν_μ) and tau neutrinos (ν_τ). Each flavour is associated with a corresponding antiparticle, called an antineutrino. However, it has been observed that neutrinos can change their flavours during their travel. That is, a neutrino which was generated with a certain flavour might end up having a different flavour after travelling some distance. This phenomenon is called **Neutrino Oscillations**. Such flavour changes require that neutrino flavours have different masses with significant mixing. This implies that neutrinos are not massless and can participate in gravitational interactions as well.[Mon15]

2.2 Leptonic Mixing

Let us assume that neutrinos have masses. Each flavor is supposed to have multiple mass eigenstates, ν_i, where $i = 1, 2, 3, etc.$, each with mass m_i. Mixing may be described with the observation that each of the three flavours of neutrinos can be expressed as a superposition of mass eigenstates. To understand this experimentally, we consider the leptonic decay

$$W^+ \to \nu_i + \bar{l_\alpha} \tag{2.1}$$

where $\bar{l_\alpha}$ is a charged lepton of flavour α. Mixing implies that every time the above decay produces a particular $\bar{l_\alpha}$, the accompanying neutrino mass eigenstate is not the same ν_i, but can be any ν_i even if the lepton has a fixed flavour. Thus, each ν_α is actually a superposition of several eigenstates ν_i 's, of which, only one state can be discerned during a single observation. Thus, we can write a flavour state $|\nu_\alpha\rangle$ in the form:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \tag{2.2}$$

The $U_{\alpha i}$'s may be written in a matrix form, called leptonic mixing matrix, assuming $i = 1, 2, 3$ and $\alpha = e, \mu, \tau$.

$$U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \tag{2.3}$$

According to Standard Model, U is Unitary.

$$UU^\dagger = U^\dagger U = I \tag{2.4}$$

Inverting (2.2),we get,

$$|\nu_i\rangle = \sum_\alpha U^*_{\alpha i} |\nu_\alpha\rangle \tag{2.5}$$

The corresponding matrix is $U^\dagger$. Assuming there are 3 mass eigenstates, the Lagrangian can be expressed in mass eigenstate (basis) terms as ,

$$\mathcal{L} = \bar{\nu} m \nu = \begin{bmatrix} \bar{\nu_1} & \bar{\nu_2} & \bar{\nu_3} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} \bar{\nu_1} & \bar{\nu_2} & \bar{\nu_3} \end{bmatrix} M_D \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \tag{2.6}$$

$\mathcal{L}$ can also be expressed in flavor basis.

$$\mathcal{L} = \begin{bmatrix} \bar{\nu_e} & \bar{\nu_\mu} & \bar{\nu_\tau} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} \bar{\nu_e} & \bar{\nu_\mu} & \bar{\nu_\tau} \end{bmatrix} M \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} \tag{2.7}$$

Clearly, M would be diagonal if there was no mixing. Using (2.2) & (2.5) in (2.7) and assuming M to be symmetric, we have,

$$M_D = U^\dagger M U \tag{2.8}$$

From (2.2) and (2.5), it is clear that the flavor α fraction in ν_i & equivalently, mass-i fraction in ν_α is $|U_{\alpha i}|^2$. This gives the probability that the neutrino will have mass m_i when a $\bar{l_\alpha}$ is observed in the decay(2.1).

2.3 Detection of oscillation

Neutrinos participate in weak interactions and very weakly interact with gravity, owing to their extremely small masses. This makes it difficult to detect them. But the charged lepton produced alongside a neutrino can be easily detected and its flavor can be determined. This, in turn, gives the flavor of the produced neutrino, say α. After travelling through a pathlength L from the source, the neutrino reacts with the detector and produces a charged lepton. Thus, we get the final flavor of the neutrino, say β. If $\alpha \neq \beta$,then the neutrino has changed its flavor during the journey, which is a quantum mechanical effect ($\nu_\alpha \to \nu_\beta$).

2.4 Oscillation Probability

Since each ν_α is a superposition of ν_i's, we have to individually add the contribution of each ν_i to find the oscillation probability of flavour conversion, $P(\nu_\alpha \to \nu_\beta)$. Hence, Amp($\nu_\alpha \to \nu_\beta$)will depend upon three factors:
Amplitude for ν_i

- When $\bar{l_\alpha}$ is produced at the source($U_{\alpha i}$)
- To propagate from the source to the detector (Prop(ν_i))
- When $\bar{l_\beta}$ is detected at the detector($U^*_{\beta i}$)

$$Amp(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i} Prop(\nu_i) U^*_{\beta i} \tag{2.9}$$

To find $Prop(\nu_i)$:
In the rest frame of the neutrino, its state vector as a function of time τ follows the Schrodinger equation (in natural units)[1],

$$i\frac{\partial}{\partial\tau}|\nu_i(\tau)\rangle = m_i\,|\nu_i(\tau)\rangle \tag{2.10}$$

whose solution is given by,

$$|\nu_i(\tau)\rangle = \exp(-im_i\tau)\,|\nu(0)\rangle \tag{2.11}$$

The amplitude of ν_i for τ_0 travel time is given by,

$$\langle\nu_i(0)|\nu_i(\tau_0)\rangle = \exp(-im_i\tau_0) \tag{2.12}$$

If τ_i be the travel time of ν_i in rest frame,

$$Prop_{Rest}(\nu_i) = \langle\nu_i(0)|\nu_i(\tau_i)\rangle = \exp(-im_i\tau_i) \tag{2.13}$$

We need Prop(ν_i) in lab frame. So we do a Lorentz tranformation to find the corresponding expression.The lab frame variables are:

- Distance between source and detector(L)
- Laboratory -frame time(t)
- Energy of mass eigenstate $\nu_i(E_i)$
- Momentum of mass eigenstate $\nu_i(p_i)$

[1] Natural units, $\hbar = c = 1$

By Lorentz Invariance,

$$m_i\tau_i = E_i t - p_i L \tag{2.14}$$

To eliminate p_i, we use the relation

$$p_i = \sqrt{(E^2 - {m_i}^2)} \simeq E - \frac{{m_i}^2}{2E} \tag{2.15}$$

(Since ${m_i}^2 \ll E^2$) Using (2.13) & (2.14),

$$m_i\tau_i \simeq Et - EL + \frac{{m_i}^2}{2E}L = E(t-L) + \frac{{m_i}^2}{2E}L \tag{2.16}$$

We have used the same E for different mass eigenstates. It can be justified as follows. Suppose two different components ν_i & ν_j, have different energies E_i & E_j. By the time they reach the detector, they have phases of e^{-iE_it} and e^{-iE_jt} respectively, where t is travel time in lab frame. Thus, the detector detects an interference caused by two components with a phase difference of $e^{-i(E_i-E_j)t}$, which vanishes for an average over time for i≠j. Thus only components with same energy are detected. The term $E(t-L)$ is common to every interfering mass eigenstate. Thus, considering only the i-dependent part,

$$Prop(\nu_i) = \exp\left(-i\frac{{m_i}^2}{2E}L\right) \tag{2.17}$$

Finally,

$$Amp(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i} \exp\Big(-i\frac{{m_i}^2}{2E}L\Big)U^*_{\beta i} \tag{2.18}$$

From (2.18),

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= |Amp(\nu_\alpha \to \nu_\beta)|^2 \\
&= \Big(\sum_i U_{\alpha i} \exp\Big(-i\frac{{m_i}^2}{2E}L\Big)U^*_{\beta i}\Big)^* \Big(\sum_j U_{\alpha j} \exp{(-i\frac{{m_j}^2}{2E}L)}U^*_{\beta j}\Big) \\
&= \sum_i \sum_j U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j} \exp\Big(i\frac{L}{2E}({m_i}^2 - {m_j}^2)\Big) \\
&= \sum_i U^*_{\alpha i}U_{\beta i}U_{\alpha i}U^*_{\beta i} + \sum_{i\neq j} U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j} \exp\Big(i\frac{L}{2E}\Delta m^2_{ij}\Big)
\end{aligned} \tag{2.19}$$

where

$$\Delta m^2_{ij} = ({m_i}^2 - {m_j}^2) \tag{2.20}$$

We derive the following quantity:

$$\begin{aligned}
\exp(iA) &= \cos A + i \sin A \\
&= 1 - 2\sin^2\frac{A}{2} + i\sin A
\end{aligned} \tag{2.21}$$

We shall expand (2.19) into four sums by using (2.21),

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= \sum_i U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} + \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \\
&\quad - 2 \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) + i \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) \\
&= P_1 + P_2 - 2P_3 + iP_4
\end{aligned} \tag{2.22}$$

We shall evaluate each part of (2.22),

$$\begin{aligned}
P_3 &= \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \\
&= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) + \sum_{i < j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\Delta m^2_{ji} \frac{L}{4E}\right) \\
&= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) + \sum_{i > j} U^*_{\alpha j} U_{\beta j} U_{\alpha i} U^*_{\beta i} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \\
&= \sum_{i > j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} + U^*_{\alpha j} U_{\beta j} U_{\alpha i} U^*_{\beta i}\right) \\
&= \sum_{i > j} \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} + (U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})^*\right) \\
P_3 &= 2 \sum_{i > j} \Re\big(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\big) \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right)
\end{aligned} \tag{2.23}$$

Now,

$$\begin{aligned}
P_4 &= \sum_{i \neq j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) \\
&= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) + \sum_{i < j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\Delta m^2_{ji} \frac{L}{2E}\right) \\
&= \sum_{i > j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) - \sum_{i > j} U^*_{\alpha j} U_{\beta j} U_{\alpha i} U^*_{\beta i} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) \\
&= \sum_{i > j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) \left[(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) - (U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})^*\right] \\
&= \sum_{i > j} \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right) (2i \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})) \\
&= 2i \sum_{i > j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right)
\end{aligned} \tag{2.24}$$

Then,

$$
\begin{aligned}
P_1 + P_2 &= \sum_i (U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}) + \sum_{i \neq j} (U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \\
&= \sum_i \sum_j U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \\
&= \sum_i (U^*_{\alpha i} U_{\beta i}) \sum_j (U_{\alpha j} U^*_{\beta j}) \\
&= \Big| \sum_i (U_{\alpha i} U^*_{\beta i}) \Big|^2
\end{aligned} \tag{2.25}
$$

Assuming 3 mass eigenstates, U and $U^\dagger$ can be written as,

$$
U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \tag{2.26}
$$

$$
U^\dagger = \begin{bmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e1} & U^*_{\mu 3} & U^*_{\tau 3} \end{bmatrix} \tag{2.27}
$$

From (2.4),

$$
UU^\dagger = \begin{bmatrix} \sum_i U_{ei} U^*_{ei} & \sum_i U_{ei} U^*_{\mu i} & \sum_i U_{ei} U^*_{\tau i} \\ \sum_i U_{\mu i} U^*_{ei} & \sum_i U_{\mu i} U^*_{\mu i} & \sum_i U_{\mu i} U^*_{\tau i} \\ \sum_i U_{\tau i} U^*_{ei} & \sum_i U_{\tau i} U^*_{\mu i} & \sum_i U_{\tau i} U^*_{\tau i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.28}
$$

Thus,

$$
\begin{aligned}
\sum_i U_{\alpha i} U^*_{\beta i} &= \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \\
&= \delta_{\alpha\beta}
\end{aligned} \tag{2.29}
$$

where $\delta_{\alpha\beta}$ is the Kronecker-delta function.
Putting (2.29) in (2.25),

$$
P_1 + P_2 = \delta_{\alpha\beta} \tag{2.30}
$$

Putting (2.23), (2.24) & (2.30) in (2.22),

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} &- 4 \sum_{i<j} \Re\Big(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\Big) \sin^2\left(\Delta m^2_{ij} \frac{L}{4E}\right) \\
&+ 2 \sum_{i<j} \Im[(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})] \sin\left(\Delta m^2_{ij} \frac{L}{2E}\right)
\end{aligned} \tag{2.31}
$$

2.5 Analysis

- To extend (2.31) to antineutrinos, we assume that CPT invariance holds true.Thus the process $\bar{\nu_\alpha} \to \bar{\nu_\beta}$ is the CPT-mirror image of $\nu_\beta \to \nu_\alpha$,

$$P(\bar{\nu_\alpha} \to \bar{\nu_\beta}) = P(\nu_\beta \to \nu_\alpha)$$

Interchanging α and β in $U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}$ gives $U^*_{\beta i}U_{\alpha i}U_{\beta j}U^*_{\alpha j}$,which is nothing but the complex conjugate of $U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}$.Thus in the expression of $P(\nu_\beta \to \nu_\alpha)$, we simply need to reverse the sign of the term containing $\Im(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})$ in (2.31). Thus,

$$\begin{aligned} P(\bar{\nu_\alpha} \to \bar{\nu_\beta}) &= P(\nu_\beta \to \nu_\alpha) \\ &= \delta_{\alpha\beta} - 4\sum_{i<j} \Re\Big(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\Big)\sin^2\left(\Delta m^2_{ij}\frac{L}{4E}\right) \\ &- 2\sum_{i<j} \Im[(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})]\sin\left(\triangle m^2_{ij}\frac{L}{2E}\right)] \end{aligned} \tag{2.32}$$

- As the probability of neutrino flavour change is a sum of sinusoidal and sine-squared functions, it necessarily oscillates with the value of $\frac{L}{E}$. Hence the term *Neutrino Oscillation* is used.

- If all neutrinos were massless, then the mass squared difference,Δm^2_{ij}=0. This reduces (2.31) and (2.32) to

$$\begin{aligned} P(\bar{\nu_\alpha} \to \bar{\nu_\beta}) &= P(\nu_\alpha \to \nu_\beta) \\ &= \delta_{\alpha\beta} \end{aligned}$$

which gives a zero probability of the event $\nu_\alpha \to \nu_\beta$ for $\alpha \neq \beta$. Hence,neutrinos observed to undergo flavour changes in vacuum implies that they are not massless and additionally, mass eigenstates are not degenerate.

- The entire calculation has been done for neutrinos travelling in vacuum. It is evident that the oscillations of neutrinos do not arise from interactions with matter, but arise from the time evolution of a neutrinos itself.

- Equation (2.31) and (2.32) contain the term Δm^2_{ij}, but do not contain the mass of each mass eigenstate explicitly. Although, we can find out the squared-mass splitting from neutrino oscillation experiments, we cannot find out the mass of each eigenstate.

- Suppose only two mass eigen states ν_1 & ν_2 were significant, with $\Delta m^2_{21} = \Delta m^2$. Correspondingly, the two flavour states are ν_e & ν_μ. We expect a corresponding 2×2 matrix U to exist, which must be unitary and has one rotation angle. Thus,

$$U = \begin{bmatrix} U_{e1} & U_{e1} \\ U_{\mu 1} & U_{\mu 2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.33}$$

and consequently,

$$U^\dagger = \begin{bmatrix} U_{e1}^* & U_{\mu 1}^* \\ U_{e2}^* & U_{\mu 2}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Using this in (2.31),

$$\begin{aligned} 4U_{\alpha 2}^* U_{\beta 2} U_{\alpha 1} U_{\beta 1}^* &= -4\sin\theta\cos\theta\cos\theta\sin\theta \\ &= -\sin^2 2\theta \end{aligned} \tag{2.34}$$

Thus, from (2.31), for $\alpha \neq \beta$,

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \delta_{\alpha\beta} - (-\sin^2 2\theta)\sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right) + 2 \times 0 \times \sin\left(\Delta m_{ij}^2 \frac{L}{2E}\right) \\ &= \sin^2 2\theta \sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right) \end{aligned} \tag{2.35}$$

The second term vanishes for a real U matrix. So, $P(\bar{\nu_\alpha} \to \bar{\nu_\beta})$ is also given by (2.35).

Chapter 3

Neutrino Oscillations in matter

Experiments to study neutrinos often involve the ones which originate at the Earth's surface, travel several kms through the bulk material of the Earth and get detected by a detector. Hence, it is very important to take the effects of matter into account. The interaction with matter affects neutrino oscillations in two ways:

1. Interactions with matter cause the neutrino to change their flavour. But according to the Standard Model, neutrino matter interactions are flavour conserving. This makes us eliminate this possibility predicted by the Standard Model.

2. On interaction with ambient particles, neutrinos can undergo forward scattering which will give rise to an extra interaction potential energy. There are two ways by which this can happen:[Kay12]

   (a) A neutrino of a particular flavour(ν_α) can exchange a W boson with it's corresponding charged lepton(l_α) only. The charged lepton available in bulk on Earth are the electrons and hence this effect is prominently observed in a ν_e and it's anti-particle, $\bar{\nu_e}$. The corresponding interaction potential will be proportional to the number density of electrons(N_e) in the bulk matter and according to the Standard Model, the potential will also be proportional to the Fermi coupling constant(G_F). From the Standard Model, we have,[Mon15]

$$V_W = \begin{cases} +\sqrt{2}G_F N_e & \text{for } \nu_e \\ -\sqrt{2}G_F N_e & \text{for } \bar{\nu_e} \end{cases} \tag{3.1}$$

   (b) According to the Standard Model, a neutrino of any flavour can exchange a Z boson with an ambient electron, proton or neutron. The couplings of the Z-boson to the electron and proton are equal and opposite. Since the bulk matter is electrically neutral , the contributions of the electron and proton via Z-boson exchange cancels out. Certainly, we end up having a flavour-independent potential, V_Z and it will be proportional to the number of neutrons per unit volume, N_n. From the Standard Model, we have,

$$V_Z = \begin{cases} -\frac{\sqrt{2}}{2} G_F N_n & \text{for } \nu \\ +\frac{\sqrt{2}}{2} G_F N_n & \text{for } \bar{\nu} \end{cases} \tag{3.2}$$

Now, consider the Schrodinger equation in the lab-frame for a neutrino travelling through matter:

$$i \frac{\partial}{\partial \tau} |\nu(t)\rangle = \mathcal{H} |\nu(t)\rangle \tag{3.3}$$

where $|\nu(t)\rangle$ is the multi-component neutrino state vector with one component for each of the neutrino flavours. For two neutrino flavours say, e and μ,

$$|\nu(t)\rangle = \begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} \tag{3.4}$$

where $f_\alpha(t)$ is the amplitude of the neutrino having a particular flavour at a time t. Therefore, $\mathcal{H}$ is a 2×2 matrix in $\nu_e - \nu_\mu$ space.

3.1 Hamiltonian in Vacuum

Let us consider the case of two neutrinos in vacuum and find out the corresponding Hamiltonian($\mathcal{H}_{vac}$). In order to find the (α, β) component of $\mathcal{H}_{vac}$, we use (2.2).

$$\begin{aligned} \langle \nu_\alpha | \mathcal{H}_{vac} | \nu_\beta \rangle &= \langle \sum_i U_{\alpha i} \nu_i | \mathcal{H}_{vac} | \sum_j U_{\beta j} \nu_j \rangle \\ &= \sum_i \langle U_{\alpha i} \nu_i | \mathcal{H}_{vac} | U_{\beta i} \nu_i \rangle \\ &= \sum_i U^*_{\alpha i} U_{\beta i} E_i \langle \nu_i | \nu_i \rangle \\ &= \sum_i U^*_{\alpha i} U_{\beta i} \sqrt{p^2 + m_i^2} \end{aligned} \tag{3.5}$$

In the third step, all terms with $\langle \nu_i | |\nu_j\rangle$ with $i \neq j$ vanish due to orthogonality.
Here, E_i is the energy of the mass eigen state ν_i. Using (3.5) and the matrix in (2.33), we evaluate each of the four terms in $\mathcal{H}_{vac}$. We denote the terms as $\mathcal{H}_{\alpha\alpha}, \mathcal{H}_{\alpha\beta}, \mathcal{H}_{\beta\alpha}$ & $\mathcal{H}_{\beta\beta}$. Also,

we use the highly relativistic approximation $\sqrt{p^2 + m_{i^2}} = \left(p + \frac{m_i^2}{2p}\right)$. Firstly,

$$\begin{aligned}
\mathcal{H}_{\alpha\alpha} &= \cos^2\theta\Big(p + \frac{m_1^2}{2p}\Big) + \sin^2\theta\Big(p + \frac{m_2^2}{2p}\Big) \\
&= (\cos^2\theta + \sin^2\theta)p + \cos^2\theta\Big(\frac{m_1^2}{2p}\Big) + \sin^2\theta\Big(\frac{m_2^2}{2p}\Big) \\
&= p + \cos^2\theta\Big(\frac{m_1^2}{2p}\Big) + (1 - \cos^2\theta)\Big(\frac{m_2^2}{2p}\Big) \\
&= p - \frac{\cos^2\theta}{2p}(m_2^2 - m_1^2) + \frac{m_2^2}{2p} \\
&= p - \frac{\Delta m^2}{2p}\cos^2\theta + \frac{m_2^2}{2p} \\
&= p + \frac{m_2^2}{2p} - \Big(\frac{2\cos^2\theta - 1}{4p}\Big)\Delta m^2 - \frac{\Delta m^2}{4p} \\
&= -\Big(\frac{2\cos^2\theta - 1}{4p}\Big)\Delta m^2 + p + \frac{2m_2^2 - \Delta m^2}{4p} \\
&= -\cos 2\theta\frac{\Delta m^2}{4p} + p + \frac{m_1^2 + m_2^2}{4p} \qquad (3.6)
\end{aligned}$$

Proceeding similarly for $\mathcal{H}_{\beta\beta}$,

$$\begin{aligned}
\mathcal{H}_{\beta\beta} &= \sin^2\theta\Big(p + \frac{m_1^2}{2p}\Big) + \cos^2\theta\Big(p + \frac{m_1^2}{2p}\Big) \\
&= p + \frac{m_1^2}{2p}\sin^2\theta + \frac{m_2^2}{2p}\cos^2\theta \\
&= p + \frac{m_1^2}{2p}(1 - \cos^2\theta) + \frac{m_2^2}{2p}\cos^2\theta \\
&= p + \frac{m_1^2}{2p} + \Big(\frac{m_2^2 - m_1^2}{2p}\Big)\Big(\frac{1 + \cos 2\theta}{2}\Big) \\
&= \cos 2\theta\Big(\frac{\Delta m^2}{4p}\Big) + p + \frac{m_1^2 + m_2^2}{4p} \qquad (3.7)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{H}_{\alpha\beta} &= \mathcal{H}_{\beta\alpha} \\
&= -\cos\theta\sin\theta\Big(p + \frac{m_1^2}{2p}\Big) + \sin\theta\cos\theta\Big(p + \frac{m_2^2}{2p}\Big) \\
&= (-\cos\theta\sin\theta + \sin\theta\cos\theta)p + \frac{\sin\theta\cos\theta}{2p}(m_2^2 - m_1^2) \\
&= \sin 2\theta\Big(\frac{\Delta m^2}{4p}\Big) \qquad (3.8)
\end{aligned}$$

Thus, from (3.6), (3.7) and (3.8), we get

$$\mathcal{H}_{vac} = \frac{\Delta m^2}{4p}\begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} + \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.9}$$

As only the relative phases of the interfering contributions matter and consequently, only the relative energies matter, we can freely subtract a multiple of the identity matrix from $\mathcal{H}_{vac}$. This does not affect the differences between the eigen values of $\mathcal{H}$. For a highly relativistic neutrino, $p \simeq E$. Thus, (3.9) becomes,

$$\mathcal{H}_{vac} = \frac{\Delta m^2}{4E}\begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \tag{3.10}$$

3.2 Hamiltonian in Matter

In order to construct the Hamiltonian in matter($\mathcal{H}_M$), two more factors(as discussed in the beginning) are considered along with $\mathcal{H}_{vac}$.

$$\mathcal{H}_M = \mathcal{H}_{vac} + V_W\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V_Z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.11}$$

The V_W term affects only ν_e's. Therefore, the upper left term corresponding to them is non-vanishing. The V_Z term affects neutrinos of all flavours. Therefore, a diagonal identity matrix is required. We can conveniently add or subtract the multiple of the identity matrix.

$$\mathcal{H}_M = \mathcal{H}_{vac} + V_W\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.12}$$

From (3.10),

$$\begin{aligned}
\mathcal{H}_M &= \frac{\Delta m^2}{4E}\begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} + \begin{bmatrix} V_W & 0 \\ 0 & -V_W \end{bmatrix} \\
&= \frac{\Delta m^2}{4E}\begin{bmatrix} -\left(\cos 2\theta - \frac{V_W/2}{\Delta m^2/4E}\right) & \sin 2\theta \\ \sin 2\theta & \left(\cos 2\theta - \frac{V_W/2}{\Delta m^2/4E}\right) \end{bmatrix} \\
&= \frac{\Delta m^2}{4E}\begin{bmatrix} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & (\cos 2\theta - x) \end{bmatrix}
\end{aligned} \tag{3.13}$$

Here,

$$x = \frac{V_W/2}{\Delta m^2/4E} = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2} \tag{3.14}$$

For simplicity, we can write (3.13) similar to (3.10). This can be done if we find X such that,

$$\cos 2\theta_M = (\cos 2\theta - x)X$$
$$\sin 2\theta_M = (\sin 2\theta)X$$
$$\Delta m_M^2 = \frac{\Delta m^2}{X}$$

Solving the above equations,

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \tag{3.15}$$

$$\Delta m_M^2 = \Delta m^2 \sqrt{\sin^2\theta + (\cos 2\theta - x)^2} \tag{3.16}$$

Thus, using (3.15) and (3.16) in (3.13),

$$\mathcal{H}_M = \frac{\Delta m_M^2}{4E}\begin{bmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{bmatrix} \tag{3.17}$$

Hence, $\mathcal{H}_M$ is similar to $\mathcal{H}_{vac}$ when Δm^2 & θ are replaced by Δm_M^2 & θ_M respectively. However, the effect values of Δm^2 & θ vary from that of vacuum in matter.

3.3 Oscillation probability in matter

For two neutrino case(from (2.2) and (2.33)),

$$|\nu_e\rangle = |\nu_1\rangle\cos\theta_M + |\nu_2\rangle\sin\theta_M$$
$$|\nu_\mu\rangle = -|\nu_1\rangle\sin\theta_M + |\nu_2\rangle\cos\theta_M \tag{3.18}$$

The eigen values of $\mathcal{H}_M$ from (3.17) are,

$$\lambda_1 = \frac{\Delta m_M^2}{4E};\lambda_2 = -\frac{\Delta m_M^2}{4E} \tag{3.19}$$

Thus, solving (3.3) with $|\nu(\tau = 0)\rangle = |\nu_e\rangle$ and using (3.18) & (3.19),

$$|\nu(t)\rangle = |\nu_1\rangle \exp\Big(-i\frac{\Delta m_M^2}{4E}t\Big)\cos\theta_M + |\nu_2\rangle\exp\Big(i\frac{\Delta m_M^2}{4E}t\Big)\sin\theta_M$$

The probability that $\nu(t)$ is detected as ν_μ,

$$\begin{aligned} P_M(\nu_e \to \nu_\mu) &= |\langle\nu_\mu|\nu(t)\rangle|^2 \\ &= \Big| -\sin\theta_M \exp\Big(-i\frac{\Delta m_M^2}{4E}t\Big)\cos\theta_M + \cos\theta_M\exp\Big(i\frac{\Delta m_M^2}{4E}t\Big)\sin\theta_M\Big|^2 \\ &= \Big|\sin\theta_M\cos\theta_M\Big(\exp\Big(i\frac{\Delta m_M^2}{4E}t\Big) - \exp\Big(-i\frac{\Delta m_M^2}{4E}t)\Big)\Big|^2 \\ &= \Big|\sin\theta_M\cos\theta_M\Big(2i\sin\frac{\Delta m_M^2}{4E}t\Big)\Big|^2 \\ &= \sin^2 2\theta_M \sin^2\Big(\frac{\Delta m_M^2 L}{4E}\Big) \end{aligned} \tag{3.20}$$

In the last step, we have replaced t with L as we have considered the highly relativistic case. Hence, the oscillation probability is similar to that of vacuum except for, Δm^2 & θ replaced by Δm_M^2 & θ_M respectively in matter.

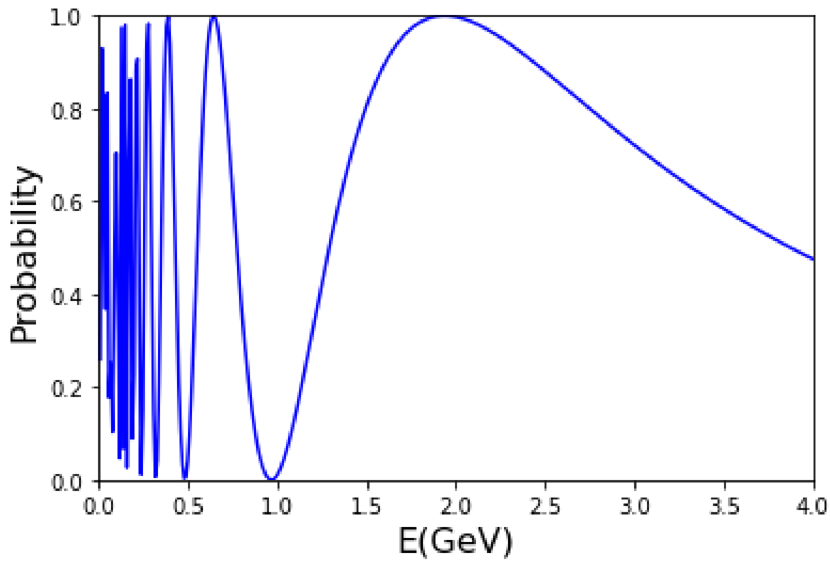


Figure 3.1: With L=1000 km,$\Delta m_{ij}^2 = 2.4 \times 10^{-3} eV^2$ at $\theta = 45°$

3.4 Discussion

- The quantity x defined by (3.14) shows the significant effects of matter on neutrino oscillations. x is proportional to E and this implies that high energy neutrinos are greatly affected by their path via matter.
- Consider a special case where θ is very small and $x \simeq \cos 2\theta$. From (3.15), $\sin^2 2\theta_M = 1$, its maximum value. Thus, matter in the path of neutrinos causes a drastic amplification of mixing angle. This is called the Mikheyev-Smirnov-Wolfenstein effect(MSW) resonance effect.
- x will have an opposite sign to it in the case of anti-neutrinos. From (3.15) & (3.16), we find that the values of Δm_M^2 and θ_M will not be the same for neutrinos and anti-neutrinos. With the difference in these values, we can differentiate between a neutrino and an anti-neutrino.

Chapter 4

See-saw mechanism

In comparison to electrons and quarks whose masses are closely related, neutrinos have very low masses which makes them unusual. In general, particles get their mass via Higg's mechanism i.e. their masses depend upon their coupling to the Higg's field. If the coupling were zero, it gives rise to zero mass or else it can result in masses of the order of Higgs or even GUT. Let us consider a possibility that two types of neutrinos exist after symmetry breaking[Kla18]. Having a zero mass or a very large mass of the symmetry breaking scale.

4.1 Math behind the mechanism

Consider a real, 2-D space with matrix(tensor) expressed in one set of orthonormal basis vectors for that space,

$$\bar{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ 0 & 100 \end{bmatrix} \tag{4.1}$$

Now, let us consider a new set of basis vectors rotated by an angle of ϕ from the original set . Then, there is a change in the components of the matrix which is given by,

$$\mathcal{M} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} 100\sin^2\phi & 100\sin\phi\cos\phi \\ 100\sin\phi\cos\phi & 100\cos^2\phi \end{bmatrix} \tag{4.2}$$

If ϕ is very small(say, $2°$),then $\cos\phi = 0.99939$ and $\sin\phi = 0.03490$.Thus,

$$\mathcal{M} = \begin{bmatrix} 0.122 & 3.488 \\ 3.488 & 99.878 \end{bmatrix} \tag{4.3}$$

We note that the lower right term is approximately the same as the original term and the upper left term is about 3 orders of magnitude smaller than the lower right term. The off

-diagonal terms equal the Geometric Mean(G.M.) of the diagonal terms ($\sqrt{(99.878)(0.122)} =$ 3.488). The off- diagonal terms are also significantly smaller than the lower right term but are not as small as the upper left term.Here, we have taken a matrix of form (4.1) and arrived at a matrix of form (4.3) by rotating it through a small angle.

4.2 Dirac and Majorana Mass terms in the Lagrangian

By theory, two distinct types of neutrino masses are allowed in the Lagrangian of Electroweak interactions :The Dirac and Majorana mass terms. Dirac particles have their own corresponding anti-particle whereas Majorana particles act as their own anti-particles. However, the two distinct mass terms do not imply anything about the nature of particles associated. As of now, we assume that both Dirac and Majorana mass terms contain only Dirac type particles.

In basic Quantum Field Theory, Dirac mass terms are given by,

$$-m_D(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L) \tag{4.4}$$

and Majorana mass terms are given by,

$$-\frac{1}{2}m_M^L(\bar{\nu}_L\nu_L^c + \bar{\nu}_L^c\nu_L) - -\frac{1}{2}m_M^R(\bar{\nu}_R\nu_R^c + \bar{\nu}_R^c\nu_R) \tag{4.5}$$

Here, L & R designate left and right handed chirality and c represents charge conjugation. So,

- ν_L destroys a LH chiral ν & creates a RH chiral $\bar{\nu}$
- $\bar{\nu_L}$ creates a LH chiral ν & destroys a RH chiral $\bar{\nu}$
- $\bar{\nu}_L^c$ destroys a LH chiral ν & creates a RH chiral $\bar{\nu}$
- ν_L^c creates a LH chiral ν & destroys a RH chiral $\bar{\nu}$

and for R subscript L and R are interchanged everywhere above. Charge conjugating a field has the same effect on a particle/anti -particle and creation /destruction as an overbar, which is effectively a complex conjugate transpose .

The first term in (4.4) destroys a RH particle and creates a LH one.Here, the weak (chiral) charge is not conserved as LH ν has $+\frac{1}{2}$ weak charge whereas the RH ν has zero weak charge. But the Lepton number is conserved, as we started with a neutrino and ended up with a neutrino.

The first term in (4.5) creates two LH νs and also does not conserve weak charge. It also does not conserve lepton number as we started with zero neutrinos and ended up with two

neutrinos.
The charge conjugation operator can be expressed as ,

$$\nu^c = C\nu = \imath\gamma^2\nu^* \tag{4.6}$$

$$\bar{\nu^c} = \nu^T C = \imath\gamma^2\nu^T \tag{4.7}$$

Writing (4.4) & (4.5) in terms of mass matrix,

$$\mathcal{L}_{massterms} = -\frac{1}{2}\begin{bmatrix}\bar{\nu}_L & \bar{\nu}^c_R\end{bmatrix}\mathcal{M}\begin{bmatrix}\nu^c_L \\ \nu_R\end{bmatrix} + h.c \tag{4.8}$$

h.c $\rightarrow$ Hermitian Conjugate of the prior term with

$$\mathcal{M} = \begin{bmatrix}m^L_M & m_D \\ m_D & m^R_M\end{bmatrix} \tag{4.9}$$

This equation is just the ν- space analog of matrix (4.3). Equation (4.8) can be rewritten as ,

$$\mathcal{L}_{massterms} = -\frac{1}{2}\begin{bmatrix}\bar{\nu}_L & \bar{\nu}^c_R\end{bmatrix}\mathcal{M}\begin{bmatrix}\nu^c_L \\ \nu_R\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\bar{\nu}^c_L & \bar{\nu}_R\end{bmatrix}\mathcal{M}\begin{bmatrix}\nu_L \\ \nu^c_R\end{bmatrix} \tag{4.10}$$

4.3 See-sawing

Suppose the Higg's or GUT breaking symmetry only gave Majorana masses to the neutrinos which means coupling was not done in a way that led to Dirac mass terms. Then, the mass matrix would be diagonal unlike (4.9).

$$\bar{\mathcal{M}} = \begin{bmatrix}m_\nu & 0 \\ 0 & M\end{bmatrix} \tag{4.11}$$

Our Lagrangian mass terms would look like,

$$\mathcal{L}_{massterms} = -\frac{1}{2}\begin{bmatrix}\bar{\nu} & \bar{N}\end{bmatrix}\bar{\mathcal{M}}\begin{bmatrix}\nu \\ N\end{bmatrix} + h.c. \tag{4.12}$$

where we have represented the fields directly coupled to the Higg's field by $\begin{bmatrix}\nu & N\end{bmatrix}^T$. That is , ν & N are mass eigen states for our neutrinos. The weak eigen states of (4.8), which are linear superpositions of ν & N interact directly via weak force and represent what we detect in weak interaction experiments.
Finding (4.11) from (4.9), is just an eigen value problem with m_ν & M as eigen values. This makes us think of two different but equivalent fields:

1. A mix of Majorana and Dirac mass terms in (4.8)

2. Pure Majorana mass terms as in (4.11), whose associated fields are represented by the column vector. $\begin{bmatrix}\nu & N\end{bmatrix}^T$

Finding $\begin{bmatrix}\nu & N\end{bmatrix}^T$ from $\begin{bmatrix}\nu_L^c & \nu_R\end{bmatrix}^T$ can be thought of as *rotating* our basis vectors in an abstract space, until there is an alignment which gives the fields vector the components $\begin{bmatrix}\nu & N\end{bmatrix}^T$. We want to know why the neutrinos have such low mass in comparison to other particles. This reason makes us put forward that the field components of the vector in (4.12) are directly coupled to the Higgs field. It works best if the mass $m_\nu = 0$, which means that there is Higgs coupling only for the N field and not the ν field. Now, (4.11) has become the analog of (4.1). If $m_\nu \neq 0$ and $m_\nu \ll M$, we would still have the question as to why one mass is much smaller than the other. Zero Higgs coupling is easier to explain as compared to extremely small coupling.

4.4 Analysis

From (4.9) & (4.11), we get the mass hierarchy

$$M \approx m_M^R \gg m_D \gg m_M^L \approx 0 \tag{4.13}$$

Also,

$$m_D^2 = m_M^R m_M^L \tag{4.14}$$

For a given value of m_D, a higher value of m_M^R means a lower value of m_M^L and vice-versa and this is why we call it the *See-Saw Mechanism*. The characteristic equation for the eigen value problem of (4.9) is given by,

$$|\mathcal{M} - \lambda I| = 0$$

$$\left| \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} m_M^L - \lambda & m_D \\ m_D & m_M^R - \lambda \end{vmatrix} = 0$$

$$(m_M^L - \lambda)(m_M^R - \lambda) - m_D^2 = 0 \tag{4.15}$$

$$\lambda^2 - (m_M^L + m_M^R)\lambda + m_M^L m_M^R - m_D^2 = 0$$

Solving the above quadratic equation, we get

$$\lambda_{1,2} = \frac{1}{2}(m_M^L + m_M^R) \pm \frac{1}{2}\sqrt{(m_M^L + m_M^R)^2 - 4(m_M^L m_M^R - m_D^2)} \tag{4.16}$$

For $\lambda_1 = m_\nu = 0$, we must have the $(-)$ sign in(4.16), then we get an expression similar to that of (4.14). Similarly, we have the plus sign for λ_2 and using λ_1, we get

$$m_M^R m_M^L = m_D^2 \tag{4.17}$$

$$\lambda_2 = M = m_M^L + m_M^R \tag{4.18}$$

Now, let the eigen vector N for λ_2 be expressed in the $\begin{bmatrix}\nu_L^c & \nu_R\end{bmatrix}^T$ basis. From (4.8) & (4.9) and with the eigen value λ_2, we get

$$\begin{aligned}(m_M^L - (m_M^R + m_M^L))\nu_L^c + m_D\nu_R = 0 \\ m_D\nu_L^c + (m_M^R - (m_M^R + m_M^L))\nu_R = 0\end{aligned} \tag{4.19}$$

This yields,

$$\nu_L^c = \frac{m_D}{m_M^R}\nu_R \tag{4.20}$$

and as an eigen vector,

$$N = \begin{bmatrix}\frac{m_D}{m_M^R}\nu_R \\ \nu_R\end{bmatrix} \tag{4.21}$$

The top component of the N matrix represents ν_L^c in terms of ν_R. Thus, N is a superposition of two fields. Including the hermitian conjugate part of (4.8) which was so far ignored , we have

$$N = (\nu_R + \nu_R^c) + \frac{m_D}{m_M^R}(\nu_L + \nu_L^c) \tag{4.22}$$

Similarly,

$$\nu = (\nu_L + \nu_L^c) - \frac{m_D}{m_M^R}(\nu_R + \nu_R^c) \tag{4.23}$$

Assuming $m_M^R \gg m_D$, from (4.18) & (4.14), we can say that N is entirely composed of ν_R & ν_R^c. It is very heavy and effectively sterile. Conversely, ν_R is said to be entirely composed of N. Similarly, ν is entirely composed of ν_L & ν_L^c. Conversely, ν_L is almost composed of weightless ν. From (4.17), we get that higher the value of m_M^R, lower is the value of m_M^L and vice versa. Thus , the name *See -saw mechanism*. From (4.22) & (4.23), for greater value of m_M^R, the more $\nu_R \to N$ & $\nu_L \to \nu$.

On the other hand, when m_M^R & m_M^L are given , their geometric mean (m_D) will be closer to the lower of the given masses.If $m_M^R = 100$, $m_M^L = 1$, then $m_D = 10$. Also, if $m_M^R \gg m_D$, from (4.17), we get $m_M^L = 0$. From (4.18), we have $m_M^R = M$. Thus, we have same the mass hierarchy as in the quick analysis method.

The Assumption $m_M^R \gg m_D$ If we start with mass matrix (4.11), with one of the fields with no coupling, then we get

$$\bar{\mathcal{M}} = \begin{bmatrix}0 & 0 \\ 0 & M\end{bmatrix} \tag{4.24}$$

and when a slight rotation is done in 2-D space of $\begin{bmatrix}\nu & N\end{bmatrix}^T$, we end up in a matrix $\bar{\mathcal{M}} = \begin{bmatrix}m_\nu & 0 \\ 0 & M\end{bmatrix}$ with $m_M^R \gg m_D$. If the see-saw mechanism exists, then we have both type mass terms of (4.4) and (4.5), and with

$$M \approx m_M^R \gg m_D > m_M^L \approx 0 \tag{4.25}$$

and for which we could have, in one scenario, Dirac neutrinos represented by ν_L and ν_R if only Dirac neutrinos exist. Alternatively, we could instead have Majorana neutrinos represented by those symbols. In either case, our interaction terms would include the symbols ν_L and ν_R, along with intermediate vector boson fields. In cases where m_D is much larger than m_M^L, the m_D mass term does not play a role in the theory at the energy levels of the present day. Thus, we effectively see the neutrinos as having mass m_M^L(The mass of Majorana mass terms in L), be they Dirac or Majorana neutrinos.

Part II

A spin one half fermionic field with mass dimension one

Chapter 5

Introduction

The era of local Quantum Field Theories started in the beginning of the twentieth century. It is a single theory that combines quantum mechanics, special relativity and classical field theory. The advent of this theory has helped us to build a successful Standard Model(SM) for particle physics. The success of SM lies in the unification of the four fundamental forces of nature: Electromagnetic, strong nuclear, weak nuclear and gravity. The Standard Model is currently able to explain all other forces except gravity. The study of gravitational force in the quantum realm is still an active area of research. Thus, quantum gravity induces non-locality.

Visible matter constitutes less than 5% of the entire matter in the universe. The rest of the universe is said to be constituted by hypothetical dark matter and dark energy. Dark matter occupies 27% and dark energy constitutes the remaining 68% roughly. It is a popular conjecture that dark matter can be realized via the study of quantum gravity. The study of Non-Standard Wigner Classes(NSWC) helps us in this pursuit. For a Standard Wigner Class, parity and charge conjugation commute for bosons and anti commute for fermions. This holds true for all the particles that have found a place in the Standard Model. Any theory which deviates this has to be non-local. In this work, we focus only on spin one half fermions which deviate the classical framework of Wigner.

What are Fermions? The word *fermion* was coined by the English theoretical physicist, Paul Dirac. Dirac had coined the word after the great Italian Physicist Enrico Fermi. In general, Fermions are particles that abide the Fermi-Dirac Statistics. These are paricles possessing half-integer spin and obey the Pauli's exclusion principle. The Pauli's exclusion principle states that no two particles can occupy the same quantum state. The class of fermions includes leptons, quarks and composite particles which include proton and neutron. These fermions have an anti-symmetric wave function.

Fermions can be classified into three types:

- Dirac fermions
- Weyl fermions
- Majorana fermions

Dirac fermions are massive fermions whereas Weyl fermions are massless fermions. The Dirac fermions can be expressed as a combination of two Weyl fermions. Lastly, the Majorana fermions constitute particles which act as their own anti particle(as discussed earlier).

Most of the fermions in the Standard Model are proven to belong to the Dirac type. So far, the Dirac equation was used to describe fermionic fields. In this work, we focus only on spin one half fermions which is said to deviate this property. Also, fermions described by the Dirac field are found to have a mass dimension of 3/2. In this work, we deal with a surprising mismatch of the mass dimension.

5.1 Scope and objective

5.1.1 Scope

Mass dimension one fermions is a flourishing domain of research in theoretical physics. They are actively studied due to potential to be candidates of dark matter.

5.1.2 Objective

- To study the group theory pertaining to high energy physics.
- To recall the implications of the fundamental Klein-Gordon and Dirac equations of quantum mechanics.
- To construct a quantum field with Elko as its expansion co-efficients.
- To derive the orthonormality and completeness relations of the Elko field.
- To study the behaviour of Elko under CPT and also its physical properties.

Chapter 6

Preliminaries

6.1 Relativistic notation

Consider two events in space-time, (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$. Let ds be the interval between two points in space-time. ds must be the same for all inertial observers .i.e, it must be invariant under Lorentz transformations and rotations. Thus, it is given by[Ryd96]

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) \tag{6.1}$$

Unlike 3-D space, this invariant interval is not positive definite. We therefore define

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{6.2}$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \tag{6.3}$$

A four-vector with upper index like x^μ, is called a *contravariant vector* and a four vector like x_μ with lower index, is called a *covariant vector*. Taking the inner product of the contravariant and covariant vector, we get an invariant(scalar).

Summation convention: An index appearing once in an upper and once in a lower position is automatically summed from 0 to 3.This is used to simplify the notation.

The relation between any contravariant and covariant vector is given by introducing the metric tensor $g_{\mu\nu}$.

$$x_\mu = g_{\mu\nu}x^\nu = g_{\mu 0}x^0 + g_{\mu 1}x^1 + g_{\mu 2}x^2 + g_{\mu 3}x^3 \tag{6.4}$$

From (6.3) and (6.4), $g_{\mu\nu}$ can be written as a diagonal matrix as follows,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.5}$$

Also,

$$g_{\mu\nu} = g^{\mu\nu} \tag{6.6}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$.

d'Alembertian Operator, $\Box$:

In particle physics, c=1 and (6.1) becomes

$$ds^2 = dx^\mu dx_\mu = dt^2 - (dx^2 + dy^2 + dz^2) \tag{6.7}$$

The differential operator is given by,

$$\partial_\mu \quad = \tfrac{\partial}{\partial x^\mu} = \left(\tfrac{1}{c}\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right)$$

So, we get

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{6.8}$$

Similarly,

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{6.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6.10}$$

From (6.8) and (6.9), we get the Lorentz invariant second-order differential operator,

$$\Box = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \tag{6.11}$$

Energy-Momentum 4-vector

The energy-momentum four-vector of a particle is given by,

$$p^\mu = \left(\tfrac{E}{c}, \quad \boldsymbol{p}\right), \qquad p_\mu = \left(\tfrac{E}{c}, \quad -\boldsymbol{p}\right) \tag{6.12}$$

The energy-momentum dispersion relation is given by,

$$E^2 = p^2 + m^2 c^4 = p^2 + m^2 \qquad \text{(Here, c=1)} \tag{6.13}$$

We derive the invariant as follows,

$$p^2 = p^\mu p_\mu = \frac{E^2}{c^2} - \boldsymbol{p} \cdot \boldsymbol{p} = m^2 \qquad \text{(from (6.13))} \tag{6.14}$$

6.2 The Klein-Gordon Equation

According to the quantum theory, the differential operators for E and $\boldsymbol{p}$ are given by,[Ryd96]

$$E \to i\hbar\frac{\partial}{\partial t}; \qquad \boldsymbol{p} \to -i\hbar\boldsymbol{\nabla} \tag{6.15}$$

Here, we arrive at an equation for a spin-zero or spinless particle. It is also called a scalar particle. Since it has no spin, it has only one component which we denote by ϕ. We obtain the wave equation by substituting the differential operators for E and $\boldsymbol{p}$ in (6.14).

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{6.16}$$

With $\hbar = c = 1$, we get,

$$(\Box + m^2)\phi = 0 \tag{6.17}$$

This is known as the *Klein-Gordon equation*. Substituting (6.15) into the non-relativistic approximation to (6.14), $E = \frac{p^2}{2m}$ yields the free particle Schrodinger equation.

$$\frac{\hbar^2}{2m}\nabla^2\phi = -i\hbar\frac{\partial\phi}{\partial t} \tag{6.18}$$

Therefore, the Schrodinger equation is the non-relativistic approximation to the Klein-Gordon equation. The Klein-Gordon equation suffers from two defects: the probability density is not positive definite and negative energy states occur. This led to the formulation of the Dirac equation.

6.3 The Dirac Equation

It is a first order equation that holds only for spin-half particles. It is derived from the transformation properties of spinors under the Lorentz group.[Ryd96]

6.3.1 The Rotation Group

In general, spatial rotation is of the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (R)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{(or)} \qquad r' = Rr \tag{6.19}$$

R is a 3×3 orthogonal matrix called the Rotation matrix. Since rotations preserve distance, we get

$$R^T R = 1 \tag{6.20}$$

These matrices form a group: if R_1 and R_2 are orthogonal. So,

$$(R_1R_2)^T R_1R_2 = R_2^T R_1^T R_1 R_2 = 1 \tag{6.21}$$

This group is denoted by $O(3)$. The matrices of rotation abot the x, y and z axes are given by,

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}; \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \tag{6.22}$$

We note that,

$$R_x(\phi)R_z(\theta) \neq R_z(\theta)R_x(\phi) \tag{6.23}$$

Hence, $O(3)$ group is *non-Abelian*. It is a *Lie group*; a continous group with an infinite number of elements, since the parameters of rotation(angles) take on a continuum of values. Corresponding to the three parameters are three generators defined by,

$$J_x = \frac{1}{i}\frac{dR_z(\theta)}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{i}\frac{dR_x(\phi)}{d\phi}\bigg|_{\phi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_y = \frac{1}{i}\frac{dR_y(\psi)}{d\psi}\bigg|_{\psi=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{6.24}$$

These are called the generators of rotation and they are hermitian. It can be verified that

$$J_xJ_y - J_yJ_x \equiv [J_x, J_y] = iJ_z \quad \text{and cyclic permutations} \tag{6.25}$$

These relations are similar to the commutation relations for components of angular momentum, with a missing factor of $\hbar$. So, angular momentum operators are the generators of rotation.

6.3.2 SU(2) group

The SU(2) group consists of 2×2 unitary matrices with unitary determinant.

$$UU^\dagger = 1; \qquad \det\ U = 1 \tag{6.26}$$

Putting

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.27}$$

the unitarity condition reads $U^\dagger = U^{-1}$. Since det U = 1, we have

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{6.28}$$

and hence $a^* = d$, $b^* = -c$.Then,

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}; \qquad det\ \ U = |a|^2 + |b|^2 = 1 \tag{6.29}$$

This is regarded as the tranformation matrix in a 2-D complex space with basic spinor $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$$\xi \to U\xi; \qquad \xi^\dagger \to \xi^\dagger U^\dagger; \qquad \xi\xi^\dagger \to U\xi\xi^\dagger U^\dagger \tag{6.30}$$

$\xi\xi^\dagger$ is a Hermitian matrix. Also, under SU(2)

$$\xi \sim \zeta\xi^*; \quad \xi^\dagger \sim (\zeta\xi)^T \quad \text{where} \quad \zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.31}$$

- An SU(2) Transformation on $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ $\equiv$O(3) transformation on $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Hence, the correspondence between (6.22) and (6.29) is,

$$U = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \leftrightarrow R = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.32}$$

Pauli matrices are regarded as generators of transformation under SU(2) group. This correspondence between SU(2) and O(3) implies that the groups have a similar structure, leading to generators obeying similar commutation relations. Hence,

$$[\sigma_x/2, \sigma_y/2] = i\frac{\sigma_z}{2}\text{and cyclic permutations} \tag{6.33}$$

The factor 1/2 indicates that the spinor rotates through half the angle that the vector rotates through.

6.3.3 The Lorentz Group

Consider two inertial frames moving with a relative speed, v along the common x axis. These two inertial frames are connected by pure *boost* Lorentz transformations. The equations are,

$$x^{0\prime} = \gamma(x^0 + \beta x^1); \quad x^{1\prime} = \gamma(\beta x^0 + x^1); \quad x^{2\prime} = x^2; \quad x^{3\prime} = x^3 \tag{6.34}$$

where, $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}; \beta = v/c; x^0 = ct; x^1 = x$. Observing that $\gamma^2 - \beta^2\gamma^2 = 1$, we put $\gamma = \cosh\phi; \gamma\beta = \sinh\phi$. Parametrising the transformation with $\tanh\phi = v/c$, we get

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{6.35}$$

The above transformation matrix is called the boost matrix B. The generator of this boost matrix along the x axis is given by,

$$K_x = \frac{1}{i}\frac{\partial \mathbf{B}}{\partial\phi}\Big|_{\phi=0} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.36}$$

Similarly, the generators of boost along y and z axes are given by,

$$K_y = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad K_z = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6.37}$$

Also, by writing the generators of rotation in 4×4 matrix notation, we get the following commutation relations,

$$[K_x, K_y] = -iK_z \quad \text{and cyclic perms;} \quad [J_x, K_x] = 0 \quad \text{etc.;} \quad [J_x, K_y] = iK_z \quad \text{and cyclic perms} \tag{6.38}$$

The pure Lorentz transformations do not form a group.

Pauli spinors under Lorentz transformations To satisfy the above commutation relations, we define

$$\mathbf{K} = \pm i\frac{\boldsymbol{\sigma}}{2} \tag{6.39}$$

There should be two types of spinor corresponding to two possible signs of $\boldsymbol{K}$. The six generators of the Lorentz group are combinely defined as,

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}); \quad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \tag{6.40}$$

From (6.25) and (6.38), their commutation relations are given by,

$$[A_i, A_j] = iA_k; \quad [B_i, B_j] = iB_k; \quad [A_i, B_j] = 0 \quad \text{(where i, j, k=x, y, z)} \tag{6.41}$$

These relations show that both **A** and **B** generate a SU(2) group each and the two groups commute. Then, the Lorentz group is nothing but SU(2)⊗SU(2) and are labelled by angular momenta (j, j′), each element corresponding to each group.
From Lorentz transformation, there are 2 different types of 2-spinors as mentioned above. Taking the parity transformation into account, it is no longer sufficient to consider 2-spinors but a 4-spinor, ψ.

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{where } \xi \text{ and } \eta \text{ are 2-spinors} \tag{6.42}$$

ψ is an irreducible representation of the Lorentz group extended by parity. So far, we have discussed a non-compact, finite-dimensional and non-unitary representation of the Lorentz group.

6.3.4 Poincaré Group

This group is an inhomogeneous Lorentz group, consisting of Lorentz boosts and rotations, and also translations in space and time. It was realised by Wigner that it is the fundamental group for particle physics. Here,

$$\xi \to \phi_R; \quad \eta \to \phi_L \tag{6.43}$$

R and L stand for right and left respectively. Under Lorentz boost, we have

$$\phi_R(\boldsymbol{p}) = \frac{E + m + \boldsymbol{\sigma} \cdot \boldsymbol{p}}{[2m(E+m)]^{1/2}} \phi_R(\mathbf{0}); \quad \phi_L(\boldsymbol{p}) = \frac{E + m + \boldsymbol{\phi} \cdot \boldsymbol{p}}{[2m(E+m)]^{1/2}} \phi_L(\mathbf{0}) \tag{6.44}$$

When a particle is at rest, one cannot define as either as right- or left-handed, so $\phi_L(\mathbf{0}) = \phi_R(\mathbf{0})$. With some algebraic steps, we get

$$\begin{pmatrix} -m & p_0 + \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ p_0 - \boldsymbol{\sigma} \cdot \boldsymbol{p} & -m \end{pmatrix} \begin{pmatrix} \phi_R(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} = 0 \tag{6.45}$$

The 4-spinor is defined as,

$$\psi(\boldsymbol{p}) = \begin{pmatrix} \phi_R(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} \tag{6.46}$$

The Dirac matrices γ^μ are given by,

$$\gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} \tag{6.47}$$

Here, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. (6.45) becomes,

$$(\gamma^\mu p_\mu - m)\psi(\boldsymbol{p}) = 0 \tag{6.48}$$

This is the Dirac equation for massive spin-half particles. For massless particles, (6.45) becomes

$$(p_0 + \boldsymbol{\sigma} \cdot \boldsymbol{p})\phi_L(\boldsymbol{p}) = 0; \quad (p_0 - \boldsymbol{\sigma} \cdot \boldsymbol{p})\phi_R(\boldsymbol{p}) = 0 \tag{6.49}$$

These are known as the *Weyl equations*, where ϕ_L and ϕ_R are Weyl spinors. For a massless particle, $p_0 = |\boldsymbol{p}|$ and (6.49) becomes,

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}\phi_L = -\phi_L; \quad \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}\phi_R = \phi_R \tag{6.50}$$

The quantity $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}$ is called *helicity*, which gives a measure of the spinor component along the direction of momentum. Thus, Weyl spinors are eigenstates of helicity. ϕ_L has negative helicity and ϕ_R has positive helicity. The Pauli matrices are given by,

$$\boldsymbol{\sigma} = \sigma_1\hat{e_1} + \sigma_2\hat{e_2} + \sigma_1\hat{e_1} \tag{6.51}$$

where,

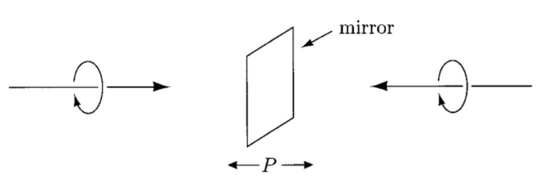
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.52}$$

Finally,
For $\mathbf{J} = \boldsymbol{\sigma}/2$ and $\mathbf{B} = \mathbf{0}$, we have the (1/2, 0) right-handed Weyl space where $\mathbf{K} = -i\boldsymbol{\sigma}/2$.
For $\mathbf{J} = \boldsymbol{\sigma}/2$ and $\mathbf{A} = \mathbf{0}$, we have the (0, 1/2) left-handed Weyl space where $\mathbf{K} = i\boldsymbol{\sigma}/2$.

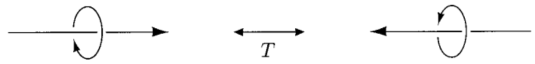
6.4 Discrete Symmetries

In addition to the continuous Lorentz trnsformations, there are two other spacetime operations that are potential symmetries of the Lagrangian: parity and time reversal.[PS05]

Parity, P Parity sends (t, **x**)→(t, -**x**),reversing the handedness of space. The operator, P should reverse the momentum of the particle without flipping its spin.



Time Reversal, T Time reversal sends (t, **x**)→(-t, **x**), interchanging the forward and backward lightcones.



Charge Conjugation This is the last of the three discrete symmetries defined to take a fermion with a given spin orientation into an antifermion with the same spin orientation. Under this operation, particles and antiparticles are interchanged.

Chapter 7

Construction the Elko Field

7.1 Introduction

The Dirac equation carries the particle-antiparticle symmetry via the operator of charge conjugation. In the Weyl space, the operator of charge conjugation is given by,[AG05a]

$$\mathcal{C} = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K \tag{7.1}$$

K⟶ Complex conjugates a spinor appearing on its right.

$$\Theta = -i\sigma_2 \tag{7.2}$$

Θ ⟶Wigner's spin half time reversal operator.
Also,

$$\Theta[\sigma/2]\Theta^{-1} = -[\sigma/2]^* \tag{7.3}$$

Using (6.52) in (7.2), we get,

$$\Theta = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.4}$$

Using (7.4) in (7.1) and also from(6.47),

$$\mathcal{C} = -\begin{pmatrix} \mathbb{O} & -\sigma_2 \\ \sigma_2 & \mathbb{O} \end{pmatrix} K = \gamma^2 K \tag{7.5}$$

The boost operator is given by,

$$B^{\pm} = \exp\Big(\pm\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\Big)$$

The boost parameter, $\boldsymbol{\varphi} = \varphi\hat{\mathbf{p}}$ in terms of energy E and momentum $\mathbf{p} = p\hat{\mathbf{p}}$ is given by, $\cosh(\varphi) = E/m$ and $\sinh(\varphi) = p/m$ where m is the mass.
We know that,

$$\boldsymbol{\varphi} = \varphi\big(\boldsymbol{p}/p\big)$$

$$\cosh\big(\varphi/2\big) = \pm\sqrt{\frac{\cosh(\varphi)+1}{2}} = \pm\sqrt{\frac{E+m}{2m}}$$

$$\sinh\big(\varphi/2\big) = \pm\sqrt{\frac{\cosh(\varphi)-1}{2}} = \pm\sqrt{\frac{E-m}{2m}}$$

$$B^{\pm} = \Big[\cosh\big(\varphi/2\big) \pm \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\sinh\big(\varphi/2\big)\Big] \tag{7.6}$$

Using the above three expressions in (7.6), we get

$$\begin{aligned} B^{\pm} &= \left[\sqrt{\frac{E+m}{2m}} \pm \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{p}\sqrt{\frac{E-m}{2m}}\right] \\ &= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1} \pm \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{\sqrt{E^2-m^2}}\sqrt{\frac{E-m}{E+m}}\right] \\ B^{\pm} &= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1} \pm \frac{p}{E+m}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\right] \end{aligned} \tag{7.7}$$

7.2 Formal Structure of Elko

Elko is an acronym of the original German term *Eigenspinoren des ladungskonjugationsoperators Elko spinors* are nothing but the Eigen spinors of the charge conjugation operator. If $\phi_L(\boldsymbol{p})$transforms as a left handed spinor, then $(\zeta\Theta)\phi_L^*(\boldsymbol{p})$ transforms as a right handed spinor. Here, ζ is an unspecified phase factor. As a consequence, these spinors belong to the $(1/2, 0) \oplus (0, 1/2)$ representation space.

$$\lambda(\boldsymbol{p}) = \begin{pmatrix} (\zeta\Theta)\phi_L^*(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} \tag{7.8}$$

These Elko spinors have real eigen values if the the unspecified phase ζ is restricted to $\pm i$. From (7.2), $\Theta = i\sigma_2$ and $\Theta^* = -i\sigma_2^*$. So, we have

$$\begin{aligned}
\Theta\Theta^* &= \sigma_2\sigma_2^* \\
&= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\Theta\Theta^* &= -\mathbb{1}
\end{aligned} \tag{7.9}$$

Also, for later use, we find,

$$\begin{aligned}
\Theta^\dagger &= (-i\sigma_2)^\dagger \\
&= +i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= +i\sigma_2 \\
\Theta^\dagger &= -\Theta
\end{aligned} \tag{7.10}$$

$$\begin{aligned}
\mathcal{C}\lambda(\boldsymbol{p}) &= \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K \begin{pmatrix} (\zeta\Theta)\phi_L^*(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} \begin{pmatrix} (\zeta\Theta)^*\phi_L(\boldsymbol{p}) \\ \phi_L^*(\boldsymbol{p}) \end{pmatrix} \\
&= \begin{pmatrix} i\Theta\phi_L^*(\boldsymbol{p}) \\ -i\Theta(\zeta\Theta)^*\phi_L(\boldsymbol{p}) \end{pmatrix} \\
\mathcal{C}\lambda(\boldsymbol{p}) &= \begin{pmatrix} i\Theta\phi_L^*(\boldsymbol{p}) \\ i(\zeta)^*\phi_L(\boldsymbol{p}) \end{pmatrix} \qquad \text{[From (7.9)]}
\end{aligned} \tag{7.11}$$

When $\zeta = +i$,

$$\mathcal{C}\lambda(\boldsymbol{p}) = \begin{pmatrix} \zeta\Theta\phi_L^*(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} = +\lambda(\boldsymbol{p})$$

The plus sign yields self-conjugate eigenspinors, $\lambda^S(\boldsymbol{p})$.
When $\zeta = -i$,

$$\mathcal{C}\lambda(\boldsymbol{p}) = -\begin{pmatrix} \zeta\Theta\phi_L^*(\boldsymbol{p}) \\ \phi_L(\boldsymbol{p}) \end{pmatrix} = -\lambda(\boldsymbol{p})$$

The minus sign yields anti self-conjugate eigenspinors, $\lambda^A(\boldsymbol{p})$. In general, we write,

$$\mathcal{C}\lambda(\boldsymbol{p}) = \pm\lambda(\boldsymbol{p}) \tag{7.12}$$

7.3 The Helicity Eigenstates

To arrive at explicit expressions for the *Elko spinors*, we consider the rest frame($\boldsymbol{p} = 0$)and find its helicity eigenstates. We take $\boldsymbol{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Let $p_1 = \sin\theta\cos\phi, p_2 = \sin\theta\sin\phi, p_3 = \cos\theta$. The helicity operator is given by,

$$\begin{aligned}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} &= \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 \qquad \text{[From (6.52)]} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} &= \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \qquad (7.13)\end{aligned}$$

The characteristic equation is given by,

$$\left|\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} - \lambda I\right| = 0$$

$$\begin{vmatrix} p_3 - \lambda & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 - \lambda \end{vmatrix} = 0$$

$$-(p_1^2 + p_2^2 + p_3^2) + \lambda^2 = 0$$

$\implies \lambda = \pm 1$

Now, we find the eigen vectors

1. When $\lambda = +1$,

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})X = \lambda X$$

$$\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$p_3 x_1 + (p_1 - ip_2)x_2 = x_1 \implies x_2 = \frac{(1-p_3)x_1}{(p_1 - ip_2)}$$

Take $x_1 = p_1 - ip_2$, then $x_2 = 1 - p_3$. So,we get

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} p_1 - ip_2 \\ 1 - p_3 \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta\cos\phi - i\sin\theta\cos\phi \\ 1 - \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta e^{-i\phi} \\ 2\sin^2(\theta/2) \end{pmatrix} \\ |\chi_1\rangle &= 2\sin(\theta/2)e^{-i\phi/2}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \qquad (7.14)\end{aligned}$$

2. When $\lambda = -1$,

$$(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}})Y = \lambda Y$$

$$\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$p_3 y_1 + (p_1 - ip_2) y_2 = -y_1 \Longrightarrow y_1 = \frac{-(p_1 - ip_2)}{(1 - p_3)} y_2$$

Take $y_2 = 1 + p_3$, then $y_1 = -(p_1 - ip_2)$. So,we get

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} -p_1 + ip_2 \\ 1 + p_3 \end{pmatrix} \\ &= \begin{pmatrix} -\sin\theta\cos\phi + i\sin\theta\cos\phi \\ 1 + \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -\sin\theta e^{-i\phi} \\ 2\cos^2(\theta/2) \end{pmatrix} \\ |\chi_2\rangle &= -2\cos(\theta/2) e^{-i\phi/2} \begin{pmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{i\phi/2} \end{pmatrix} \end{aligned} \tag{7.15}$$

Finally, the helicity eigenstates with adopted phases include,

$$\phi_L^+(\mathbf{0}) = \sqrt{m} e^{i\mathfrak{v}_1} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \tag{7.16}$$

$$\phi_L^-(\mathbf{0}) = \sqrt{m} e^{i\mathfrak{v}_2} \begin{pmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{i\phi/2} \end{pmatrix} \tag{7.17}$$

Here, $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathfrak{R}$. We set $\mathfrak{v}_1 = 0, \mathfrak{v}_2 = 0$. These are the locality phase factors chosen to remove the non-locality. Thus, (7.16) and (7.17) become,

$$\phi_L^+(\mathbf{0}) = \sqrt{m} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \tag{7.18}$$

$$\phi_L^-(\mathbf{0}) = \sqrt{m} \begin{pmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{i\phi/2} \end{pmatrix} \tag{7.19}$$

The eigenvalue equation can be written as,

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}[\phi_L^{\pm}(\mathbf{0})] = \pm \phi_L^{\pm}(\mathbf{0}) \tag{7.20}$$

Useful Identities

1.

$$(\phi_L^+(\mathbf{0}))^\dagger(\phi_L^+(\mathbf{0})) = m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}\cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2}\end{pmatrix} = m \quad (7.21)$$

Similarly,

$$(\phi_L^+(\mathbf{0}))^\dagger(\phi_L^-(\mathbf{0})) = m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}\sin(\theta/2)e^{-i\phi/2} \\ -\cos(\theta/2)e^{i\phi/2}\end{pmatrix} = 0 \quad (7.22)$$

Calculations can be carried out for other possible combinations of the helicity eigenstates. Finally, we can generalise the above identities as,

$$(\phi_L^\alpha(\mathbf{0}))^\dagger(\phi_L^\beta(\mathbf{0})) = m\delta_{\alpha\beta} \tag{7.23}$$

α and β are either $+$ or $-$. Here, $\delta_{\alpha\beta}$ is a Kronecker-delta function.

2.

$$\begin{aligned}(\phi_L^+(\mathbf{0}))^\dagger\sigma_2(\phi_L^+(\mathbf{0}))^* &= m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2}\end{pmatrix} \\ &= m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}-i\sin(\theta/2)e^{-i\phi/2} \\ i\cos(\theta/2)e^{i\phi/2}\end{pmatrix} \\ &= 0\end{aligned} \tag{7.24}$$

3.

$$\begin{aligned}(\phi_L^+(\mathbf{0}))^\dagger\sigma_2(\phi_L^-(\mathbf{0}))^* &= m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\sin(\theta/2)e^{i\phi/2} \\ -\cos(\theta/2)e^{-i\phi/2}\end{pmatrix} \\ &= m\begin{pmatrix}\cos(\theta/2)e^{i\phi/2} & \sin(\theta/2)e^{-i\phi/2}\end{pmatrix}\begin{pmatrix}i\cos(\theta/2)e^{-i\phi/2} \\ i\sin(\theta/2)e^{i\phi/2}\end{pmatrix} \\ &= im\end{aligned} \tag{7.25}$$

A similar calculation can be done for other possible combinations of helicity eigenstates. Generalising the above two relations, we get,

$$(\phi_L^\pm(\mathbf{0}))^\dagger\sigma_2(\phi_L^\pm(\mathbf{0}))^* = 0 \tag{7.26}$$

$$(\phi_L^\pm(\mathbf{0}))^\dagger\sigma_2(\phi_L^\mp(\mathbf{0}))^* = \pm im \tag{7.27}$$

7.4 Explicit Form of Elko

To construct the Elko spinors, we need to choose global phases for each of the helicity eigen states,

$$\lambda^S_+(\mathbf{0}) = e^{i\xi_1}\begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix}, \quad \lambda^S_-(\mathbf{0}) = e^{i\xi_2}\begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix}; \tag{7.28}$$

$$\lambda^A_+(\mathbf{0}) = e^{i\xi_3}\begin{pmatrix} -i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix}, \quad \lambda^A_-(\mathbf{0}) = e^{i\xi_4}\begin{pmatrix} -i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix} \tag{7.29}$$

with $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathfrak{R}$. We set $\xi_1 = \xi_2 = \xi_3 = 0$ and $\xi_4 = \pi$. In the above expressions, we set $\zeta = +i$ for self-conjugate spinors and $\zeta = -i$ for anti self-conjugate spinors. Then, (7.28) and (7.29) become,

$$\lambda^S_+(\mathbf{0}) = \begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix} = \lambda^S_{\{-,+\}}(\mathbf{0}), \quad \lambda^S_-(\mathbf{0}) = \begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix} = -\lambda^S_{\{+,-\}}(\mathbf{0}); \tag{7.30}$$

$$\lambda^A_+(\mathbf{0}) = \begin{pmatrix} -i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix} = -\lambda^A_{\{+,-\}}(\mathbf{0}), \quad \lambda^A_-(\mathbf{0}) = \begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ -\phi^+_L(\mathbf{0})\end{pmatrix} = -\lambda^A_{\{-,+\}}(\mathbf{0}) \tag{7.31}$$

The general expression for self-conjugate spinors is given by,

$$\lambda^S_\pm(\mathbf{0}) = \begin{pmatrix} i\Theta[\phi^\pm_L(\mathbf{0})]^* \\ \phi^\pm_L(\mathbf{0})\end{pmatrix} = \pm\lambda^S_{\{\mp,\pm\}}(\mathbf{0}) \tag{7.32}$$

The general expression for anti self-conjugate spinors is given by,

$$\lambda^A_\pm(\mathbf{0}) = \pm\begin{pmatrix} -i\Theta[\phi^\mp_L(\mathbf{0})]^* \\ \phi^\mp_L(\mathbf{0})\end{pmatrix} = -\lambda^A_{\{\pm,\mp\}}(\mathbf{0}) \tag{7.33}$$

Generalising (7.32) and (7.33), we get,

$$\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) = \begin{pmatrix} \zeta\Theta[\phi^\pm_L(\mathbf{0})]^* \\ \phi^\pm_L(\mathbf{0})\end{pmatrix} \tag{7.34}$$

Boosted Spinor: We get the boosted spinors by using (7.7) and (7.34),

$$\begin{aligned}\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) &= B^\pm\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) \\ &= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1} \pm \frac{p}{E+m}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\right]\begin{pmatrix} \zeta\Theta[\phi^\pm_L(\mathbf{0})]^* \\ \phi^\pm_L(\mathbf{0})\end{pmatrix}\end{aligned} \tag{7.35}$$

Using (7.20) and (7.43) in (7.35), we get,

$$\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1 \mp \frac{p}{E+m}\right)\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) \tag{7.36}$$

In the Massless Limit:

(a)

$$\sqrt{\frac{E+m}{2m}}\left(1-\frac{p}{E+m}\right)=\sqrt{\frac{E+m}{2m}}-\frac{p}{\sqrt{2m(E+m)}}=\frac{E+m-p}{\sqrt{2m(E+m)}} \tag{7.37}$$

From (6.13), when m→ 0, $E=p$. So, (7.37) becomes,

$$\begin{aligned}\lim_{m\to 0}\frac{E+m-p}{\sqrt{2m(E+m)}} &= \lim_{m\to 0}\frac{m}{\sqrt{2m(E+m)}}\\ &= \lim_{m\to 0}\frac{1}{\sqrt{2}}\sqrt{\frac{m}{E+m}}\\ &= \lim_{m\to 0}\sqrt{\frac{m}{2E}}\left(1+\frac{m}{E}\right)^{-1/2}\\ &= \lim_{m\to 0}\sqrt{\frac{m}{2E}}\left(1-\frac{m}{2E}\right)\\ \lim_{m\to 0}\frac{E+m-p}{\sqrt{2m(E+m)}} &= 0\end{aligned} \tag{7.38}$$

Thus, from (7.36) and (7.38), $\lambda_{\{-,+\}}(\boldsymbol{p})$ identically vanishes in the massless limit.

(b)

$$\begin{aligned}\sqrt{\frac{E+m}{2m}}\left(1+\frac{p}{E+m}\right) &= \sqrt{\frac{E+m}{2m}}+\frac{p}{\sqrt{2m(E+m)}}\\ &= \frac{E+m+p}{\sqrt{2m(E+m)}}\end{aligned} \tag{7.39}$$

From (6.13), when m→ 0, $E=p$. So, (7.39) becomes,

$$\begin{aligned}\lim_{m\to 0}\frac{E+m+p}{\sqrt{2m(E+m)}} &= \lim_{m\to 0}\frac{2E+m}{\sqrt{2m(E+m)}}\\ &= \lim_{m\to 0}\sqrt{\frac{2}{m}}\left(\frac{E}{\sqrt{(E+m)}}\right)+\sqrt{\frac{m}{2(E+m)}}\\ &= \lim_{m\to 0}\sqrt{\frac{2}{mE}}\frac{1}{\sqrt{1+(\frac{m}{E})}}+\sqrt{\frac{m}{2E}}\frac{1}{\sqrt{1+(\frac{m}{E})}}\\ &= \lim_{m\to 0}\left(1-\frac{m}{2E}\right)\left(\sqrt{\frac{2}{mE}}+\sqrt{\frac{m}{2E}}\right)\\ \lim_{m\to 0}\frac{E+m+p}{\sqrt{2m(E+m)}} &= \sqrt{\frac{2}{mE}}\end{aligned} \tag{7.40}$$

Thus, from(7.36) and (7.40), $\lambda_{\{+,-\}}(\boldsymbol{p})$ does not vanish identically in the massless limit.

Now, complex conjugating (7.20),

$$\boldsymbol{\sigma}^* \cdot \hat{\boldsymbol{p}}[\phi_L^{\pm}(\mathbf{0})]^* = \pm[\phi_L^{\pm}(\mathbf{0})]^* \tag{7.41}$$

From (7.4),

$$\Theta^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\Rightarrow \Theta^{-1} = -\Theta$. Also, from (7.3), we have $\Theta[\sigma]\Theta^{-1} = -\sigma^*$. Using these expressions in (7.41), we get,

$$\Theta\boldsymbol{\sigma}\Theta^{-1} \cdot \hat{\boldsymbol{p}}[\phi_L^{\pm}(\mathbf{0})]^* = \mp[\phi_L^{\pm}(\mathbf{0})]^*$$

$$-\Theta\boldsymbol{\sigma}\Theta \cdot \hat{\boldsymbol{p}}[\phi_L^{\pm}(\mathbf{0})]^* = \mp[\phi_L^{\pm}(\mathbf{0})]^*$$

(or)

$$\Theta^{-1}\boldsymbol{\sigma}\Theta \cdot \hat{\boldsymbol{p}}[\phi_L^{\pm}(\mathbf{0})]^* = \mp[\phi_L^{\pm}(\mathbf{0})]^* \tag{7.42}$$

Left multiplying both sides of (7.42) by Θ and moving Θ through $\hat{\boldsymbol{p}}$,

$$\Theta\Theta^{-1}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}\Theta[\phi_L^{\pm}(\mathbf{0})]^* = \mp\Theta[\phi_L^{\pm}(\mathbf{0})]^*$$

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}\Theta[\phi_L^{\pm}(\mathbf{0})]^* = \mp\Theta[\phi_L^{\pm}(\mathbf{0})]^* \tag{7.43}$$

It physically implies that $\Theta[\phi_L^{\pm}(\mathbf{0})]^*$ has opposite helicity to that of $\phi_L^{\pm}(\mathbf{0})$. Since $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}$ commutes with $B^{\pm}$ and this result holds for all $\boldsymbol{p}$.

Thus, Elko are not single helicity objects and they cannot be eigenspinors of the helicity operator.

Chapter 8

The Elko Dual Spinor and Properties

8.1 Defining the Elko Dual

The Elko dual is uniquely defined as,

$$\bar{\lambda}^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) := \pm i\big[\lambda^{S/A}_{\{\pm,\mp\}}\big]^{\dagger}\gamma^0 \tag{8.1}$$

With respect to the Dirac dual, the Elko dual has an imaginary bi-orthogonal norm.[AG05a]

8.2 Bi-orthonormality Relations

1.

$$\begin{aligned}\bar{\lambda}^{S}_{\{-,+\}}(\mathbf{p})\lambda^{S}_{\{-,+\}}(\mathbf{p}) &= +i\big[\lambda^{S}_{\{+,-\}}(\mathbf{p})\big]^{\dagger}\gamma^0\lambda^{S}_{\{-,+\}}(\mathbf{p})\\ &= i\left(\frac{E+m}{2m}\right)\left(1-\Big(\frac{p}{E+m}\Big)^2\right)\begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix}^{\dagger}\begin{pmatrix}\mathbb{O} & \mathbb{1}\\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix}\end{aligned} \tag{8.2}$$

$$\begin{aligned}\begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix}^{\dagger}\begin{pmatrix}\mathbb{O} & \mathbb{1}\\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix} &= \begin{pmatrix}-i([\phi^-_L(\mathbf{0})]^*)^{\dagger}\Theta^{\dagger} & ([\phi^-_L(\mathbf{0})]^*)^{\dagger}\end{pmatrix}\begin{pmatrix}\phi^+_L(\mathbf{0}) \\ i\Theta[\phi^+_L(\mathbf{0})]^*\end{pmatrix}\\ &= (-i([\phi^-_L(\mathbf{0})]^*)^{\dagger}\Theta^{\dagger}\phi^+_L(\mathbf{0}) + ([\phi^-_L(\mathbf{0})]^*)^{\dagger} i\Theta[\phi^+_L(\mathbf{0})]^*)\\ &= (([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2^*\phi^+_L(\mathbf{0}) + ([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2[\phi^+_L(\mathbf{0})]^*) \quad \text{[From(7.10)]}\end{aligned}$$

$$\begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0})\end{pmatrix}^{\dagger}\begin{pmatrix}\mathbb{O} & \mathbb{1}\\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0})\end{pmatrix} = -2im \qquad \text{[From(7.27)]} \tag{8.3}$$

Then,

$$\left(\frac{E+m}{2m}\right)\left(1-\left(\frac{p}{E+m}\right)^2\right) = \frac{(E+m)^2-p^2}{2m(E+m)}$$
$$= \frac{E^2-p^2+m^2+2mE}{2m(E+m)} \qquad \text{[From(6.13)]}$$
$$= \frac{2m^2+2mE}{2m(E+m)}$$
$$\left(\frac{E+m}{2m}\right)\left(1-\left(\frac{p}{E+m}\right)^2\right) = 1 \tag{8.4}$$

Substituting (8.3) and (8.4) in (8.2), we get,

$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 2m \tag{8.5}$$

2.

$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = +i\left[\lambda^S_{\{+,-\}}(\mathbf{p})\right]^\dagger\gamma^0\lambda^S_{\{+,-\}}(\mathbf{p})$$
$$= i\left(\frac{E+m}{2m}\right)\left(1+\frac{p}{E+m}\right)^2\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix}^\dagger\begin{pmatrix}\mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix} \tag{8.6}$$

$$\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix}^\dagger\begin{pmatrix}\mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix} = \begin{pmatrix} -i([\phi_L^-(\mathbf{0})]^*)^\dagger\Theta^\dagger & ([\phi_L^-(\mathbf{0})]^*)^\dagger\end{pmatrix}\begin{pmatrix}\phi_L^-(\mathbf{0}) \\ i\Theta[\phi_L^-(\mathbf{0})]^*\end{pmatrix}$$
$$= (-i([\phi_L^-(\mathbf{0})]^*)^\dagger\Theta^\dagger\phi_L^-(\mathbf{0}) + ([\phi_L^-(\mathbf{0})]^*)^\dagger i\Theta[\phi_L^-(\mathbf{0})]^*)$$
$$= (([\phi_L^-(\mathbf{0})]^*)^\dagger\sigma_2^*\phi_L^-(\mathbf{0}) + ([\phi_L^-(\mathbf{0})]^*)^\dagger\sigma_2[\phi_L^-(\mathbf{0})]^*) \qquad \text{[From(7.1}$$
$$\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix}^\dagger\begin{pmatrix}\mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0})\end{pmatrix} = 0 \qquad \text{[From(7.26)]} \tag{8.7}$$

Substituting (8.7) in (8.6), we get,

$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 0 \tag{8.8}$$

3.

$$\bar{\lambda}^S_{\{+,-\}}(\mathbf{p})\lambda^S_{\{-,+\}}(\mathbf{p}) = -i\left[\lambda^S_{\{-,+\}}(\mathbf{p})\right]^\dagger\gamma^0\lambda^S_{\{-,+\}}(\mathbf{p})$$
$$= -i\left(\frac{E+m}{2m}\right)\left(1+\frac{p}{E+m}\right)^2\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0})\end{pmatrix}^\dagger\begin{pmatrix}\mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0})\end{pmatrix} \tag{8.9}$$

$$\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}^\dagger \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix} = \begin{pmatrix} -i([\phi_L^+(\mathbf{0})]^*)^\dagger\Theta^\dagger & ([\phi_L^+(\mathbf{0})]^*)^\dagger \end{pmatrix} \begin{pmatrix} \phi_L^+(\mathbf{0}) \\ i\Theta[\phi_L^+(\mathbf{0})]^* \end{pmatrix}$$
$$= (-i([\phi_L^+(\mathbf{0})]^*)^\dagger\Theta^\dagger\phi_L^+(\mathbf{0}) + ([\phi_L^+(\mathbf{0})]^*)^\dagger i\Theta[\phi_L^+(\mathbf{0})]^*)$$
$$= (([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2^*\phi_L^+(\mathbf{0}) + ([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2[\phi_L^+(\mathbf{0})]^*) \quad \text{[From(7.10)]}$$
$$\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}^\dagger \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix} = 0 \quad \text{[From(7.26)]} \tag{8.10}$$

Substituting (8.10) in (8.9), we get,

$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 0 \tag{8.11}$$

4.

$$\bar{\lambda}^S_{\{+,-\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = -i\left[\lambda^S_{\{-,+\}}(\mathbf{p})\right]^\dagger\gamma^0\lambda^S_{\{+,-\}}(\mathbf{p})$$
$$= -i\left(\frac{E+m}{2m}\right)\left(1-\left(\frac{p}{E+m}\right)^2\right)\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}^\dagger \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}$$
$$= -i\begin{pmatrix} -i([\phi_L^+(\mathbf{0})]^*)^\dagger\Theta^\dagger & ([\phi_L^+(\mathbf{0})]^*)^\dagger \end{pmatrix} \begin{pmatrix} \phi_L^-(\mathbf{0}) \\ i\Theta[\phi_L^-(\mathbf{0})]^* \end{pmatrix}$$
$$= -i(-i([\phi_L^+(\mathbf{0})]^*)^\dagger\Theta^\dagger\phi_L^-(\mathbf{0}) + ([\phi_L^+(\mathbf{0})]^*)^\dagger i\Theta[\phi_L^-(\mathbf{0})]^*)$$
$$= -i(([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2^*\phi_L^-(\mathbf{0}) + ([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2[\phi_L^-(\mathbf{0})]^*) \quad \text{[From(7.10)]}$$
$$\bar{\lambda}^S_{\{+,-\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 2m \quad \text{[From(7.27)]} \tag{8.12}$$

Summarising the bi-orthonormality conditions for the self-conjugate spinors, we have,

$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 2m \quad ; \quad \bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 0$$
$$\bar{\lambda}^S_{\{-,+\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 0 \quad ; \quad \bar{\lambda}^S_{\{+,-\}}(\mathbf{p})\lambda^S_{\{+,-\}}(\mathbf{p}) = 2m$$

Thus, the generalised orthonormal condition for self-conjugate spinors is given by,

$$\bar{\lambda}^S_\alpha(\mathbf{p})\lambda^S_{\alpha'}(\mathbf{p}) = +2m\delta_{\alpha\alpha'}\delta_{SI} \tag{8.13}$$

Similar exercise can be carried out to arrive at the bi-orthonormality conditions for the anti self-conjugate spinors and the corresponding generalised orthonormal relation is given by,

$$\bar{\lambda}^A_\alpha(\mathbf{p})\lambda^A_{\alpha'}(\mathbf{p}) = -2m\delta_{\alpha\alpha'}\delta_{AI} \tag{8.14}$$

In label(8.13) and (8.14), α & α' take values of either $\{-,+\}$ or $\{+,-\}$ and $I \in \{S, A\}$.

8.3 Spin Sums and Completeness Relation

Spin Sums Here, α ranges over two possibilities: $\{-,+\},\{+,-\}$

$$\sum_{\alpha}\lambda^S_{\alpha}(\boldsymbol{p})\bar{\lambda}^S_{\alpha}(\boldsymbol{p}) = \lambda^S_{\{-,+\}}(\boldsymbol{p})\bar{\lambda}^S_{\{-,+\}}(\boldsymbol{p}) + \lambda^S_{\{+,-\}}(\boldsymbol{p})\bar{\lambda}^S_{\{+,-\}}(\boldsymbol{p}) \tag{8.15}$$

The calculations are carried out in a step-wise manner as follows.

1.

$$\begin{aligned}
\lambda^S_{\{-,+\}}(\boldsymbol{p})\bar{\lambda}^S_{\{-,+\}}(\boldsymbol{p}) &= i\lambda^S_{\{-,+\}}(\boldsymbol{p})[\lambda^S_{\{+,-\}}(\boldsymbol{p})]^{\dagger}\gamma^0 \\
&= i\left(\frac{E+m}{2m}\right)\left(1-\left(\frac{p}{E+m}\right)^2\right)\lambda^S_{\{-,+\}}(\mathbf{0})[\lambda^S_{\{+,-\}}(\mathbf{0})]^{\dagger}\gamma^0 \\
&= i\lambda^S_{\{-,+\}}(\mathbf{0})[\lambda^S_{\{+,-\}}(\mathbf{0})]^{\dagger}\gamma^0 \qquad \text{(From (8.4))}
\end{aligned} \tag{8.16}$$

Firstly,

$$\begin{aligned}
[\lambda^S_{\{+,-\}}(\mathbf{0})]^{\dagger} &= \begin{pmatrix} i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0}) \end{pmatrix}^{\dagger} \\
&= \begin{pmatrix} -i([\phi^-_L(\mathbf{0})]^*)^{\dagger}\Theta^{\dagger} & (\phi^-_L(\mathbf{0}))^{\dagger} \end{pmatrix} \\
&= \begin{pmatrix} ([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2 & (\phi^-_L(\mathbf{0}))^{\dagger} \end{pmatrix}
\end{aligned} \tag{8.17}$$

Substituting (8.17), we get

$$\begin{aligned}
\lambda^S_{\{-,+\}}(\boldsymbol{p})[\lambda^S_{\{+,-\}}(\boldsymbol{p})]^{\dagger}\gamma^0 &= \begin{pmatrix} i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0}) \end{pmatrix}\begin{pmatrix} ([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2 & (\phi^-_L(\mathbf{0}))^{\dagger} \end{pmatrix}\begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \\
&= \begin{pmatrix} \sigma_2[\phi^+_L(\mathbf{0})]^*(\phi^-_L(\mathbf{0}))^{\dagger} & \sigma_2[\phi^+_L(\mathbf{0})]^*([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2 \\ \phi^+_L(\mathbf{0})(\phi^-_L(\mathbf{0}))^{\dagger} & \phi^+_L(\mathbf{0})([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2 \end{pmatrix}
\end{aligned} \tag{8.18}$$

Now, we calculate each of the elements in the matrix (8.18).

(a)

$$\sigma_2[\phi^+_L(\mathbf{0})]^*(\phi^-_L(\mathbf{0}))^{\dagger} = im\begin{pmatrix} -\sin^2(\theta/2) & \sin(\theta/2)\cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2)\cos(\theta/2)e^{i\phi} & -\cos^2(\theta/2) \end{pmatrix} \tag{8.19}$$

(b)

$$\sigma_2[\phi^+_L(\mathbf{0})]^*([\phi^-_L(\mathbf{0})]^*)^{\dagger}\sigma_2 = im\begin{pmatrix} i\sin(\theta/2)\cos(\theta/2) & i\sin^2(\theta/2)e^{-i\phi} \\ -i\cos^2(\theta/2)e^{i\phi} & -i\sin(\theta/2)\cos(\theta/2) \end{pmatrix} \tag{8.20}$$

(c)

$$\phi_L^+(\mathbf{0})(\phi_L^-(\mathbf{0}))^\dagger = m\begin{pmatrix} \sin(\theta/2)\cos(\theta/2) & -\cos^2(\theta/2)e^{-i\phi} \\ \sin^2(\theta/2)e^{i\phi} & -\sin(\theta/2)\cos(\theta/2) \end{pmatrix} \tag{8.21}$$

(d)

$$\phi_L^+(\mathbf{0})([\phi_L^-(\mathbf{0})]^*)^\dagger\sigma_2 = \begin{pmatrix} -i\cos^2(\theta/2) & -i\sin(\theta/2)\cos(\theta/2))e^{-i\phi} \\ -i\sin(\theta/2)\cos(\theta/2))e^{i\phi} & -i\sin^2(\theta/2) \end{pmatrix} \tag{8.22}$$

For convenience, we take $(\theta/2) = x$. Putting together (8.19), (8.20), (8.21) and (8.22) in (8.18), (8.16) becomes

$$m\begin{pmatrix} \sin^2 x & -\sin x\cos xe^{-i\phi} & -i\sin x\cos x & -i\sin^2 xe^{-i\phi} \\ -\sin x\cos xe^{i\phi} & \cos^2 x & i\cos^2 xe^{i\phi} & i\sin x\cos x \\ i\sin x\cos x & -i\cos^2 xe^{-i\phi} & \cos^2 x & \sin x\cos xe^{-i\phi} \\ i\sin^2 xe^{i\phi} & -i\sin x\cos x & \sin x\cos xe^{i\phi} & \sin^2 x \end{pmatrix} \tag{8.23}$$

2.

$$\begin{aligned} \lambda^S_{\{+,-\}}(\boldsymbol{p})\bar{\lambda}^S_{\{+,-\}}(\boldsymbol{p}) &= -i\lambda^S_{\{+,-\}}(\boldsymbol{p})[\lambda^S_{\{-,+\}}(\boldsymbol{p})]^\dagger\gamma^0 \\ &= -i\left(\frac{E+m}{2m}\right)\left(1-\left(\frac{p}{E+m}\right)^2\right)\lambda^S_{\{+,-\}}(\mathbf{0})[\lambda^S_{\{-,+\}}(\mathbf{0})]^\dagger\gamma^0 \\ &= -i\lambda^S_{\{-,+\}}(\mathbf{0})[\lambda^S_{\{+,-\}}(\mathbf{0})]^\dagger\gamma^0 \qquad \text{(From (8.4))} \end{aligned} \tag{8.24}$$

Firstly,

$$\begin{aligned} [\lambda^S_{\{-,+\}}(\mathbf{0})]^\dagger &= \begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}^\dagger \\ &= \begin{pmatrix} -i([\phi_L^+(\mathbf{0})]^*)^\dagger\Theta^\dagger & (\phi_L^+(\mathbf{0}))^\dagger \end{pmatrix} \\ &= \begin{pmatrix} ([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 & (\phi_L^+(\mathbf{0}))^\dagger \end{pmatrix} \end{aligned} \tag{8.25}$$

Substituting (8.17), we get

$$\begin{aligned} \lambda^S_{\{+,-\}}(\boldsymbol{p})[\lambda^S_{\{-,+\}}(\boldsymbol{p})]^\dagger\gamma^0 &= \begin{pmatrix} i\Theta[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}\begin{pmatrix} ([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 & (\phi_L^+(\mathbf{0}))^\dagger \end{pmatrix}\begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \\ &= \begin{pmatrix} \sigma_2[\phi_L^-(\mathbf{0})]^*(\phi_L^+(\mathbf{0}))^\dagger & \sigma_2[\phi_L^-(\mathbf{0})]^*([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 \\ \phi_L^-(\mathbf{0})(\phi_L^+(\mathbf{0}))^\dagger & \phi_L^-(\mathbf{0})([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 \end{pmatrix} \end{aligned} \tag{8.26}$$

Now, we calculate each of the elements in the matrix (8.26).

(a)

$$\sigma_2[\phi_L^-(\mathbf{0})]^*(\phi_L^+(\mathbf{0}))^\dagger = im\begin{pmatrix} \cos^2(\theta/2) & \sin(\theta/2)\cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2)\cos(\theta/2)e^{i\phi} & \sin^2(\theta/2) \end{pmatrix} \tag{8.27}$$

(b)

$$\sigma_2[\phi_L^-(\mathbf{0})]^*([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 = im\begin{pmatrix} i\sin(\theta/2)\cos(\theta/2) & -i\cos^2(\theta/2)e^{-i\phi} \\ i\sin^2(\theta/2)e^{i\phi} & -i\sin(\theta/2)\cos(\theta/2) \end{pmatrix} \tag{8.28}$$

(c)

$$\phi_L^-(\mathbf{0})(\phi_L^+(\mathbf{0}))^\dagger = m\begin{pmatrix} \sin(\theta/2)\cos(\theta/2) & \sin^2(\theta/2)e^{-i\phi} \\ -\cos^2(\theta/2)e^{i\phi} & -\sin(\theta/2)\cos(\theta/2) \end{pmatrix} \tag{8.29}$$

(d)

$$\phi_L^-(\mathbf{0})([\phi_L^+(\mathbf{0})]^*)^\dagger\sigma_2 = im\begin{pmatrix} \sin^2(\theta/2) & -\sin(\theta/2)\cos(\theta/2)e^{-i\phi} \\ -\sin(\theta/2)\cos(\theta/2)e^{i\phi} & \cos^2(\theta/2) \end{pmatrix} \tag{8.30}$$

For convenience, we take $(\theta/2) = x$. Putting together (8.27), (8.28), (8.29) and (8.30) in (8.26), (8.24) becomes

$$m\begin{pmatrix} \cos^2 x & \sin x\cos xe^{-i\phi} & i\sin x\cos x & -i\cos^2 xe^{-i\phi} \\ \sin x\cos xe^{i\phi} & \sin^2 x & i\sin^2 xe^{i\phi} & -i\sin x\cos x \\ -i\sin x\cos x & -i\sin^2 xe^{-i\phi} & \sin^2 x & -\sin x\cos xe^{-i\phi} \\ i\cos^2 xe^{i\phi} & i\sin x\cos x & -\sin x\cos xe^{i\phi} & \cos^2 x \end{pmatrix} \tag{8.31}$$

From (8.23) and (8.31), (8.15) becomes

$$\begin{aligned}
\sum_\alpha \lambda_\alpha^S(\boldsymbol{p})\bar{\lambda}_\alpha^S(\boldsymbol{p}) &= m\begin{pmatrix} 1 & 0 & 0 & -ie^{-i\phi} \\ 0 & 1 & ie^{i\phi} & 0 \\ 0 & -ie^{-i\phi} & 1 & 0 \\ ie^{i\phi} & 0 & 0 & 1 \end{pmatrix} \\
&= m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + m\begin{pmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \end{pmatrix} \\
\sum_\alpha \lambda_\alpha^S(\boldsymbol{p})\bar{\lambda}_\alpha^S(\boldsymbol{p}) &= m[\mathbb{I} + \mathcal{G}(\boldsymbol{p})]
\end{aligned} \tag{8.32}$$

Similar calculations are carried out for anti self-conjugate spinors and the spin sum is found to be,

$$\sum_{\alpha} \lambda_{\alpha}^{A}(\boldsymbol{p})\bar{\lambda}_{\alpha}^{A}(\boldsymbol{p}) = -m[\mathbb{I} - \mathcal{G}(\boldsymbol{p})] \tag{8.33}$$

It is found that $\mathcal{G}$ is an odd function of $\boldsymbol{p}$. So, $\mathcal{G}(\boldsymbol{p}) = -\mathcal{G}(-\boldsymbol{p})$.

Subtracting (8.33) from (8.32), we get

$$\sum_{\alpha} \left[\lambda_{\alpha}^{S}(\boldsymbol{p})\bar{\lambda}_{\alpha}^{S}(\boldsymbol{p}) - \lambda_{\alpha}^{A}(\boldsymbol{p})\bar{\lambda}_{\alpha}^{A}(\boldsymbol{p})\right] = 2m\mathbb{I}$$

Thus, the completeness relation is given by

$$\frac{1}{2m}\sum_{\alpha} \left[\lambda_{\alpha}^{S}(\boldsymbol{p})\bar{\lambda}_{\alpha}^{S}(\boldsymbol{p}) - \lambda_{\alpha}^{A}(\boldsymbol{p})\bar{\lambda}_{\alpha}^{A}(\boldsymbol{p})\right] = \mathbb{I} \tag{8.34}$$

It shows the necessity of the anti self-conjugate spinors.

Chapter 9

Behaviour of Elko under $\mathcal{CPT}$

9.1 Parity, $\mathcal{P}$

The parity operator is given by, $\mathcal{P} = m^{-1}\gamma^\mu p_\mu$. Elko spinors transform under parity as follows,

$$\begin{aligned}\mathcal{P}\lambda_+^S(\mathbf{p}) &= m^{-1}\gamma^\mu p_\mu \sqrt{\frac{E+m}{2m}}\left[1 - \frac{p}{E+m}\right]\lambda_+^S(\mathbf{0}) \\ &= \sqrt{\frac{E+m}{2m}}\left[1 - \frac{p}{E+m}\right] m^{-1}\left[E\gamma^0 + p\begin{pmatrix}\mathbb{O} & \boldsymbol{\sigma}\cdot\hat{p} \\ \boldsymbol{\sigma}\cdot\hat{p} & \mathbb{O}\end{pmatrix}\right]\lambda_+^S(\mathbf{0}) \qquad \text{(From (6.47))}\end{aligned} \tag{9.1}$$

$$\begin{aligned}\begin{pmatrix}\mathbb{O} & \boldsymbol{\sigma}\cdot\hat{p} \\ \boldsymbol{\sigma}\cdot\hat{p} & \mathbb{O}\end{pmatrix}\lambda_+^S(\mathbf{0}) &= \begin{pmatrix}\mathbb{O} & \boldsymbol{\sigma}\cdot\hat{p} \\ \boldsymbol{\sigma}\cdot\hat{p} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0})\end{pmatrix} \\ &= \begin{pmatrix}\phi_L^+(\mathbf{0}) \\ i\Theta[\phi_L^+(\mathbf{0})]^*\end{pmatrix} \qquad \text{(From (7.20) \& (7.43))} \\ &= \begin{pmatrix}\mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O}\end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0})\end{pmatrix}\end{aligned}$$

$$\begin{pmatrix}\mathbb{O} & \boldsymbol{\sigma}\cdot\hat{p} \\ \boldsymbol{\sigma}\cdot\hat{p} & \mathbb{O}\end{pmatrix}\lambda_+^S(\mathbf{0}) = \gamma^0\lambda_+^S(\mathbf{0}) \tag{9.2}$$

Also, we have the identity,

$$\gamma^0\lambda_+^S(\mathbf{0}) = +i\lambda_-^S(\mathbf{0}) \tag{9.3}$$

From (9.2) and (9.3), (9.1) becomes,

$$\begin{aligned}\mathcal{P}\lambda_+^S(\mathbf{p}) &= \sqrt{\frac{E+m}{2m}}\left(1-\frac{p}{E+m}\right)m^{-1}(E+p)\gamma^0\lambda_+^S(\mathbf{0})\\ &= +i\sqrt{\frac{E+m}{2m}}\left(1-\frac{p}{E+m}\right)m^{-1}(E+p)\lambda_-^S(\mathbf{0})\end{aligned}\tag{9.4}$$

From (6.13), we simplify

$$\begin{aligned}\Big(1-\frac{p}{E+m}\Big)(E+p) &= \frac{(E^2-p^2)+m(E+p)}{E+m}\\ &= m\Big(1+\frac{p}{E+m}\Big)\end{aligned}\tag{9.5}$$

Using (9.5) and (7.36) in (9.4), we have,

$$\mathcal{P}\lambda_+^S(\mathbf{p}) = +i\sqrt{\frac{E+m}{2m}}\left(1+\frac{p}{E+m}\right)\lambda_-^S(\mathbf{0}) = +i\lambda_-^S(\mathbf{p})\tag{9.6}$$

By similar calculations,

$$\mathcal{P}\lambda_-^S(\mathbf{p}) = -i\lambda_+^S(\mathbf{p})\tag{9.7}$$

$$\mathcal{P}\lambda_+^A(\mathbf{p}) = -i\lambda_-^A(\mathbf{p})\tag{9.8}$$

$$\mathcal{P}\lambda_-^A(\mathbf{p}) = +i\lambda_+^A(\mathbf{p})\tag{9.9}$$

Generalising,

$$\mathcal{P}\lambda_\pm^S(\mathbf{p}) = \pm i\lambda_\mp^S(\mathbf{p}) \quad ; \mathcal{P}\lambda_\pm^A(\mathbf{p}) = \mp i\lambda_\mp^A(\mathbf{p})\tag{9.10}$$

Further, applying the parity operator twice, we get,

$$\mathcal{P}^2[\lambda_\pm^S(\mathbf{p})] = \mathcal{P}[\pm i\lambda_\mp^S(\mathbf{p})] = \lambda_\pm^S(\mathbf{p})\tag{9.11}$$

Similarly,

$$\mathcal{P}^2[\lambda_\pm^A(\mathbf{p})] = \mathcal{P}[\mp i\lambda_\mp^A(\mathbf{p})] = \lambda_\pm^A(\mathbf{p})\tag{9.12}$$

The above expressions imply that, $\mathcal{P}^2 = -\mathbb{I}_4$. Here, $\mathbb{I}_4$ is a 4×4 identity matrix.

9.2 Time Reversal, $\mathcal{T}$

The time reversal operator is given by, $\mathcal{T} = i\gamma^5\mathcal{C}$. We know that, $\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}$.

$$\begin{aligned}
\mathcal{T}\lambda_+^S(p^\mu) &= i\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}\mathcal{C}\lambda_+^S(p^\mu) \\
&= i\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}\lambda_+^S(p^\mu) \\
&= i\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix} \\
&= i\begin{pmatrix} i\Theta[\phi_L^+(\mathbf{0})]^* \\ -\phi_L^+(\mathbf{0}) \end{pmatrix} \\
\mathcal{T}\lambda_+^S(p^\mu) &= i\lambda_-^A(p^\mu) \qquad (9.13)
\end{aligned}$$

By similar calculations,

$$\mathcal{T}\lambda_-^S(p^\mu) = -i\lambda_+^A(p^\mu) \qquad (9.14)$$

$$\mathcal{T}\lambda_+^A(p^\mu) = i\lambda_-^S(p^\mu) \qquad (9.15)$$

$$\mathcal{T}\lambda_-^A(p^\mu) = -i\lambda_+^S(p^\mu) \qquad (9.16)$$

The above expressions imply that $\mathcal{T}^2 = -\mathbb{I}_4$.

9.3 Charge Conjugation, $\mathcal{C}$ and Parity, $\mathcal{P}$

From (7.12) and (9.6), we have

$$\mathcal{CP}[\lambda_+^S(\mathbf{p})] = +i\mathcal{C}[\lambda_-^S(\mathbf{p})] = +i\lambda_-^S(\mathbf{p}) \qquad (9.17)$$

and

$$\mathcal{PC}[\lambda_+^S(\mathbf{p})] = \mathcal{P}[\lambda_+^S(\mathbf{p})] = +i\lambda_-^S(\mathbf{p}) \qquad (9.18)$$

Then, from (9.17) and (9.18), $[\mathcal{C}, \mathcal{P}] = 0$. Similar calculations can be carried out for the other Elko spinors and find that the commutation relation is valid for all $\lambda(\mathbf{p})$.

9.4 Charge Conjugation, $\mathcal{C}$ and Time Reversal, $\mathcal{T}$

$$\mathcal{CT}[\lambda_+^S(\mathbf{p})] = \mathcal{C}[+i\lambda_-^A(\mathbf{p})] = -i\lambda_-^A(\mathbf{p}) \qquad (9.19)$$

and

$$\mathcal{TC}[\lambda_+^S(\mathbf{p})] = \mathcal{T}[\lambda_+^S(\mathbf{p})] = i\lambda_-^A(\mathbf{p}) \tag{9.20}$$

Then, from (9.19) and (9.20), $[\mathcal{C}, \mathcal{T}] = 0$. Their commuting property can also be checked with the other Elko spinors.

9.5 Parity, $\mathcal{P}$ and Time Reversal, $\mathcal{T}$

$$\mathcal{PT}[\lambda_+^S(\mathbf{p})] = \mathcal{P}[+i\lambda_-^A(\mathbf{p})] = \lambda_+^A(\mathbf{p}) \tag{9.21}$$

and

$$\mathcal{TP}[\lambda_+^S(\mathbf{p})] = \mathcal{T}[+i\lambda_-^S(\mathbf{p})] = \lambda_+^A(\mathbf{p}) \tag{9.22}$$

Then, from (9.21) and (9.22), $[\mathcal{P}, \mathcal{T}] = 0$. This commutation relation can be found to be valid for all $\lambda(\mathbf{p})$.
From the above analysis for Elko, $[\mathcal{C}, \mathcal{P}] = 0$, $[\mathcal{P}, \mathcal{T}] = 0$, $[\mathcal{C}, \mathcal{T}] = 0$. This proves our claim that Elko belongs to a NSWC. Also, Wigner's expectation $(\mathcal{CPT})^2 = -\mathbb{I}$ is confirmed and reconcile with Weinberg's observation due to Elko's dual helicity nature.[Wei95]

Chapter 10

Physical Properties of Elko

10.1 The Field

An Elko-based quantum field is a spin one half matter field with mass dimension one. It has well-defined CPT properties and can be defined as[AG05b]

$$\eta(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\boldsymbol{p})}} \sum_{\beta} \left[c_{\beta}(\boldsymbol{p}) \lambda^{S}_{\beta}(\boldsymbol{p}) e^{-ip_{\mu}x^{\mu}} + c^{\dagger}_{\beta}(\boldsymbol{p}) \lambda^{A}_{\beta}(\boldsymbol{p}) e^{ip_{\mu}x^{\mu}} \right] \tag{10.1}$$

The creation and annihilation operators, $c^{\dagger}_{\beta}(\boldsymbol{p})$ and $c_{\beta}(\boldsymbol{p})$ are expected to follow the below anti-commutation relations,

$$\{c_{\beta}(\boldsymbol{p}), c^{\dagger}_{\beta'}(\boldsymbol{p'})\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p'}) \delta_{\beta\beta'} \tag{10.2}$$

$$\{c^{\dagger}_{\beta}(\boldsymbol{p}), c^{\dagger}_{\beta'}(\boldsymbol{p'})\} = \{c_{\beta}(\boldsymbol{p}), c_{\beta'}(\boldsymbol{p'})\} = 0 \tag{10.3}$$

Its Elko dual $\eta(\bar{x})$ is defined as,

$$\bar{\eta}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\boldsymbol{p})}} \sum_{\beta} \left[c^{\dagger}_{\beta}(\boldsymbol{p}) \bar{\lambda}^{S}_{\beta}(\boldsymbol{p}) e^{ip_{\mu}x^{\mu}} + c_{\beta}(\boldsymbol{p}) \bar{\lambda}^{A}_{\beta}(\boldsymbol{p}) e^{-ip_{\mu}x^{\mu}} \right] \tag{10.4}$$

10.2 The Elko Propagator

The amplitude for a positive-energy self conjugate particle to propagate from x to x' is $\langle s(x') | s(x) \rangle$ The state $|s(x)\rangle$ contains one positive-energy self conjugate particle of mass m. Therefore, the covariant amplitude is defined as,

$$\mathcal{Q}_{x \to x'} = \varpi \left\langle \quad | \, \eta(x') \bar{\eta}(x) \, | \quad \right\rangle, \quad \varpi \in \mathbb{C} \tag{10.5}$$

Since fermionic amplitudes are anti-symmetric under exchange $x \leftrightarrow x'$, the total covariant amplitude is given by

$$\varpi \left\langle \quad | \, \eta(x') \bar{\eta}(x) \, | \quad \right\rangle \theta(t' - t) - \varpi \left\langle \quad | \, \bar{\eta}(x) \eta(x') \, | \quad \right\rangle \theta(t - t') \tag{10.6}$$

On evaluating, we get

$$\mathcal{Q}_{x\to x'} = \varpi \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\boldsymbol{p})}\sum_{\beta}\left[\theta(t'-t)\lambda^S_\beta(\boldsymbol{p})\bar{\lambda}^S_\beta(\boldsymbol{p})e^{-ip_\mu(x'^\mu-x^\mu)}-\theta(t-t')\lambda^A_\beta(\boldsymbol{p})\bar{\lambda}^A_\beta(\boldsymbol{p})e^{ip_\mu(x'^\mu-x^\mu)}\right] \tag{10.7}$$

From (8.32) and (8.33),

$$\mathcal{Q}_{x\to x'} = \varpi \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\boldsymbol{p})}\sum_{\beta}\left[\theta(t'-t)(\mathbb{I}+\mathcal{G}(\phi))e^{-ip_\mu(x'^\mu-x^\mu)}+\theta(t-t')(\mathbb{I}-\mathcal{G}(\phi))e^{ip_\mu(x'^\mu-x^\mu)}\right] \tag{10.8}$$

Let $\boldsymbol{p}\to-\boldsymbol{p}$ in the second term and using (8.3),

$$\mathcal{Q}_{x\to x'} = \varpi \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\boldsymbol{p})}\sum_{\beta}\left[\theta(t'-t)(\mathbb{I}+\mathcal{G}(\phi))e^{-iE(\boldsymbol{p})(t'-t)+i\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})}+\theta(t-t')(\mathbb{I}+\mathcal{G}(\phi))e^{+iE(\boldsymbol{p})(t'-t)+i\boldsymbol{p}\cdot(\boldsymbol{x}'-} \tag{10.9}$$

The integral representation of the heaviside step function is given by,

$$\theta(t'-t) = \lim_{\epsilon\to 0^+}\int \frac{d\omega}{2\pi i}\frac{e^{i\omega(t'-t)}}{\omega-i\epsilon}; \qquad \theta(t-t') = \lim_{\epsilon\to 0^+}\int \frac{d\omega}{2\pi i}\frac{e^{i\omega(t-t')}}{\omega-i\epsilon} \tag{10.10}$$

Inserting (10.10) into (10.9),

$$\mathcal{Q}_{x\to x'} = -i\varpi \lim_{\epsilon\to 0^+}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2mE(\boldsymbol{p})}\int\frac{d\omega}{2\pi}\times\Big[\frac{(\mathbb{I}+\mathcal{G}(\phi))}{\omega-i\epsilon}\Big(e^{i(\omega-E(\boldsymbol{p}))(t'-t)}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})}+e^{-i(\omega-E(\boldsymbol{p}))(t'-t)}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\Big) \tag{10.11}$$

In the first integral, $\omega\to p_0 = -(\omega-E(\boldsymbol{p}))$ and in the second integral, $\omega\to p_0=(\omega-E(\boldsymbol{p}))$. On substituting (10.11) becomes,

$$\mathcal{Q}_{x\to x'} = -i\varpi \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{1}{2mE(\boldsymbol{p})}e^{-ip_\mu(x'^\mu-x^\mu)}\left[\frac{\mathbb{I}+\mathcal{G}(\phi)}{E(\boldsymbol{p})-p_0-i\epsilon}+\frac{\mathbb{I}+\mathcal{G}(\phi)}{E(\boldsymbol{p})+p_0-i\epsilon}\right] \tag{10.12}$$

Neglecting the terms of order ϵ^2, (10.12) becomes

$$\mathcal{Q}_{x\to x'} = \int \frac{d^4p}{(2\pi)^4}e^{-ip_\mu(x'^\mu-x^\mu)}\left[i\varpi\frac{\mathbb{I}+\mathcal{G}(\phi)}{p_\mu p^\mu-m^2+i\epsilon}\right] \tag{10.13}$$

When there is no ϕ dependence,

$$\mathcal{Q}_{x\to x'} = \int \frac{d^4p}{(2\pi)^4}e^{-ip_\mu(x'^\mu-x^\mu)}\left[i\varpi\frac{\mathbb{I}}{p_\mu p^\mu-m^2+i\epsilon}\right] \tag{10.14}$$

There is no preferred spacetime point. Integrating (10.14) over all possible $x-x'$ and setting the result to unity,

$$(2\pi)^4\int \frac{d^4p}{(2\pi)^4}\delta^4(p^\mu)\left[i\varpi\frac{\mathbb{I}}{p_\mu p^\mu-m^2+i\epsilon}\right] = 1 \tag{10.15}$$

That is,

$$i\varpi\frac{\mathbb{I}}{-m^2+i\epsilon}=\mathbb{I} \tag{10.16}$$

Taking $\epsilon \to 0$ yields,

$$\varpi = im^2 \tag{10.17}$$

$$\mathcal{Q}_{x\to x'} = -m\int\frac{d^4p}{(2\pi)^4}e^{-ip_\mu(x'^\mu - x^\mu)}\left[\frac{m\mathbb{I}}{p_\mu p^\mu - m^2 + i\epsilon}\right] \tag{10.18}$$

Therefore, the propagator for the new theory is,

$$\begin{aligned}\mathcal{S}^{Elko}_{FD}(x',x) &:= -\frac{1}{m^2}\mathcal{Q}_{x\to x'}\\ &= \int\frac{d^4p}{(2\pi)^4}e^{-ip_\mu(x'^\mu - x^\mu)}\left[\frac{\mathbb{I}}{p_\mu p^\mu - m^2 + i\epsilon}\right]\end{aligned} \tag{10.19}$$

In the absence of a preferred direction, the Elko-propagator is identical to that of a scalar Klein-Gordon field. It is this circumstance that endows the $\eta(x)$ with mass dimension one. For a massive field of Lorentz transformation type (A,B), the mass dimension of the field is given by $1+A+B$. So, for field of types (1/2,0) and (0, 1/2), the mass dimensionality has to be 3/2. But for the Elko field which also transforms as (1/2,0)$\oplus$(0,1/2), the mass dimensionality is 1 and not 3/2. The reason lies in the non-locality of the field.[Wei95]

Alternate way of showing mass dimension mismatch In natural units($\hbar = c = 1$), Action becomes dimensionless. The action integral for Dirac field is given by[SR09]

$$\mathcal{S}_{Dirac} = \int d^4x(\bar{\psi}\gamma^\mu\partial_\mu\psi - m_D(\bar{\psi}\psi))$$

d^4x has a dimension of L^4, ∂_μ has a dimension of L^{-1}. Thus, the wavefunction must have a dimension of $L^{-3/2}$. As we know, length is inversely proportional to mass. Hence the **Dirac field has a mass dimension of 3/2**. The action integral for Elko field is given by,

$$\mathcal{S}_{Elko} = \int d^4x(\partial_\mu\bar{\lambda}\partial^\mu\lambda - m^2\lambda\bar{\lambda})$$

In a similar fashion, the Elko spinor is found to have a dimension of $(1/L)$. And consequently, the **Elko field has mass dimension of one**.

Chapter 11

Summary

This project required some basics of quantum field theory and fundamental understanding of the identity- agnostic neutrinos. A considerable amount of time has been spent in studying the rich history behind the emergence of neutrino physics. The oscillation probabilities were studied in detail through quantum mechanics. In the second semester, the construction, behaviour under CPT and the physical properties of the Elko field have been worked out with mathematics as the key.

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