THERMODYNAMICS OF BLACK HOLES IN ANTI-DE SITTER SPACETIME

Project Report

submitted in partial fulfillment of the requirement for the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

KEERTHANA U

Reg No:176PH016

DEPARTMENT OF PHYSICS
NATIONAL INSTITUTE OF TECHNOLOGY
KARNATAKA,SURATHKAL,MANGALORE-575025

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Under the guidance of

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DECLARATION

I hereby declare that the report of the P.G. Project Work entitled "THERMODYNAMICS OF BLACK HOLES IN ANTI-DE SITTER SPACETIME" which is submitted to National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators.

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CERTIFICATE

This is to certify that the project entitled "THERMODYNAMICS OF BLACK HOLES IN ANTI-DE SITTER SPACETIME" is an authenticated record of work carried out by KEERTHANA U ,Reg.No:176PH016 in partial fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology, Karnataka, during the period 2018-2019.

Dr.DEEPAK VAID
Project Advisor

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ABSTRACT

Anti-de Sitter space is one of the fascinating model because of its closed time-like curve. This property can be studied using conformal diagram. The conformal diagram of Reissner-Nordstörm black hole is studied.Investigation of P-V diagram shows that it behaves like a Vander Waals fluid. The study of thermodynamics of Bardeen black hole tells that it shows first order phase transition. The first order phase transition is studied using C_p v/s r_+ , G v/s r_+ graphs. The study of critical exponents shows that the bardeen black hole also behaves like Vander Waals fluid.

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Chapter 1

Introduction

Most of the cosmological models are based on cosmological principle which says that the universe is homogeneous and isotropic on a large scale. One such solution of Einstein equation is anti-de Sitter(AdS) space which is a maximally symmetric space-time with negative curvature.

1.1 Isotropy and Homogeneity

Isotropic space is a space which looks same independent of the direction. In turn, isotropy is invariant under rotation.

$$f(x') = f(\mathbf{R}x) = f(x)$$

i.e., the value of rotated coordinates take exactly same as that of the initial coordinates.
Homogeneity means the space is same throughout the manifold. That means the property measured at point p and q are same. In turn, homogeneity is translational invariant.
i.e., for any arbitrary value x' the function can be written as

$$f(x) = f(x + x')$$

The existence of isotropy and homogeneity are often leads to the spherically symmetry.Using the isotropic coordinates, the general spherically symmetric spacetime metric can be written as[Car14],

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 (dr^2 + r^2 d\Omega^2)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of 2-sphere, f and g are functions of radial coordinate. The range be $-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \theta \leq \pi$ and $-\pi \leq \phi \leq \pi$.

1.2 Metric of Anti-de Sitter space

For a anti-de Sitter space of radius l, the hyperboloid of D+1 dimension is given by

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + ... + X_D^2 = -l^2$$

For D+1 dimensional hyperbolic space,the metric can be written as,

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + ... + dX_D^2$$

Considering five-dimensional flat manifold embedded in a hyperboloid,

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 = -l^2 \tag{1.1}$$

The space is $\mathbf{R}^{3,2}$. Here the Cartesian coordinates X_0 and X_1 are having timelike dimension and X_2,X_3 and X_4 are having spacelike dimension.And the metric be

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + +dX_4^2 \tag{1.2}$$

Parametric equations by introducing the new coordinates (t, ρ,θ,ϕ) be[Car14],

$$\begin{aligned}
X_0 &= l \sin t \cosh \rho \\
X_1 &= l \cos t \cosh \rho \\
X_2 &= l \sinh \rho \cos \theta \\
X_3 &= l \sinh \rho \sin \theta \cos \phi \\
X_4 &= l \sinh \rho \sin \theta \sin \phi
\end{aligned}$$

such that the equations satifiy (1.1). To obtain the metric,

$$dX_0 = l(\cos t \cosh \rho dt + \sin t \sinh \rho d\rho)$$

$$dX_1 = l(-\sin t \cosh \rho dt + \cos t \sinh \rho d\rho)$$

$$dX_2 = l(\cosh \rho \cos \theta d\rho - \sinh \rho \sin \theta d\theta)$$

$$dX_3 = l(\cosh \rho \sin \theta \cos \phi d\rho + \sinh \rho \cos \theta \cos \phi d\theta - \sinh \rho \sin \theta \sin \phi d\phi)$$

$$dX_4 = l(\cosh \rho \sin \theta \sin \phi d\rho + \sinh \rho \cos \theta \sin \phi d\theta + \sinh \rho \sin \theta \cos \phi d\phi)$$

are substituted in (1.2).

$$\begin{aligned}
ds^2 = &-[(\cos t \cosh \rho dt + \sin t \sinh \rho d\rho)]^2 - [l(-\sin t \cosh \rho dt + \cos t \sinh \rho d\rho)]^2 \\
&+ [l(\cosh \rho \cos \theta d\rho - \sinh \rho \sin \theta d\theta)]^2 + [l(\cosh \rho \sin \theta \cos \phi d\rho + \sinh \rho \cos \theta \cos \phi d\theta \\
&- \sinh \rho \sin \theta \sin \phi d\phi)]^2 + [l(\cosh \rho \sin \theta \sin \phi d\rho + \sinh \rho \cos \theta \sin \phi d\theta + \sinh \rho \sin \theta \cos \phi d\phi)]^2
\end{aligned}$$

On rearrangement,

$$ds^2 = -(l^2 \cosh^2 \rho dt^2 + l^2 \sinh^2 \rho d\rho^2) + (l^2 \cosh^2 \rho d\rho^2 + l^2 \sinh^2 \rho d\theta^2 + l^2 \sinh^2 \rho \sin^2 \theta d\phi^2)$$

$$ds^2 = l^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2) \tag{1.3}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of 2-sphere.
For a constant proper time,the metric can be written as

$$ds^2 = l^2(d\rho^2 + \sinh^2 \rho d\Omega_2^2)$$

This metric is similar to the Lobechevski metric which is having negative constant curvature.Thus the space can also be termed as pseudosphere.

1.3 Global Property of AdS space

As mentioned earlier, AdS space is a pseudosphere for a constant time. Now replacing t by t+2π in the coordinate, (t, r, θ, ϕ) and $(t + 2\pi, r, \theta, \phi)$ represent the same place in pseudosphere. That represents that the time coordinate is periodic. Then the range of time coordinate can be given as $-\pi \leq t \leq \pi$. In other words, for a constant (r, θ, ϕ), timelike curve froms a circle. Here the hyperbolic space has the topology of $\mathbf{R}^3$ and the timelike curve is of $\mathbf{S}^1$. Hence the AdS space is the product of $\mathbf{S}^1$ and $\mathbf{R}^3$ i.e., $\mathbf{S}^1 \times \mathbf{R}^3$.
This periodic identification of timelike curve can be avoided by considering $\mathbf{R}^{3,2}$ space. The range of time coordinate can be taken as $-\infty \leq t \leq \infty$. This has the topology of $\mathbf{R}^1$. Thus the entire AdS spacetime topology be $\mathbf{R}^4$.This space time is called *universal covering space of AdS space*. This makes the AdS space unique.[Sok16]

1.4 Thermodynamics

Any system that has certain pressure, volume and temperature can be studied by calculating its thermal properties. The state of the system can be examined using equation of state which is basically a thermodynamic equation that gives the relationship between pressure, volume and temperature. If we plot these variables along x,y,z axes, the equation defines the surface of state. By keeping temperature constant, P-V can be measured. There exists inflection point P-V diagram above which the isotherms are continuous curves, below which there maxima and minima in the curves. The behaviour of the system near the critical point can also be studied using critical point where the quantity can be described by power law.

System stability can be studied using entropy, specific heat capacity, Gibbs free energy etc. Entropy is the quantity that measures the degree of disorderness. As the entropy decreases, the system is more ordered, thus the system is more stable. Specific heat capacity is the amount of heat required to raise 1kg of solid by 1 degree celsius. Specific heat capacity has to be positive for system to be stable. Negative specific heat represents the unstability of the system. Gibbs free energy is the available energy. Smaller the free energy value, the system is spontaneous and stable. Gibbs free energy place a main role in the study of phase transition which is discussed in the next section.

1.5 Phase Transition

A phase is a state of matter in thermal equilibrium. The transition of a matter from one state to another state with respect to the first is phase transition. The same matter will have different thermal properties in different phases. The phase transition boundary can be described using pressure, volume and temperature. The diagram which shows the relation between pressure, volume and temperature is used to study the phase transition. Phase transition can be first order and second order phase transition depending on the lowest order of the differential of G which shows the discontinuity.

1.5.1 First Order Phase Transition

In the first order phase transition, there will be discontinuity in the internal energy, volume and specific heat capacity at constant pressure. In the process of transition, heat is either absorbed or evolved.
According to phase equilibrium condition, the Gibbs free energy in two phases are equal at a particular pressure and temperature. Let the Gibbs free energy of first and second phase be G_α(P,T) and G_β(P,T),

$$
\begin{aligned}
G_\alpha(P,T) &= G_\beta(P,T) \\
G_{\alpha\beta}(P,T) &= G_\alpha(P,T) - G_\beta(P,T) = 0
\end{aligned}
$$

where $G_{\alpha\beta}$(P,T) is the differences between the Gibbs free energies in two phases.
Then the small change in pressure and temperature with respect to the original temperature and pressure should also satisfy phase equilibrium condition.

$$
\begin{aligned}
&G_{\alpha\beta}(P+dP,T+dT) = 0 \\
&G_{\alpha\beta}(P+dP,T+dT) = G_{\alpha\beta}(P,T) + \frac{\partial G_{\alpha\beta}(P,T)}{\partial P}\Big|_T dP + \frac{\partial G_{\alpha\beta}(P,T)}{\partial T}\Big|_P dT = 0
\end{aligned}
$$

Since $G_{\alpha\beta}$(P,T) is always 0 at equilibrium.

$$
\frac{\partial G_{\alpha\beta}(P,T)}{\partial P}\Big|_T dP + \frac{\partial G_{\alpha\beta}(P,T)}{\partial T}\Big|_P dT = 0
$$

where,

$$
\frac{\partial G_{\alpha\beta}(P,T)}{\partial P}\Big|_T = V \qquad\qquad \frac{\partial G_{\alpha\beta}(P,T)}{\partial T}\Big|_P = -S
$$

1.5.2 Second Order Phase Transition

In the second order phase transition, internal energy, volume change smoothly. But the second order derivative of Gibbs free energy is discontinuous where as its first order derivatives are continuous.
Carrying out the same procedure gives,

$$
\frac{\partial^2 G_{\alpha\beta}(P,T)}{\partial P^2}(dP)^2 + 2\frac{\partial^2 G_{\alpha\beta}}{\partial P\partial T}dPdT + \frac{\partial^2 G_{\alpha\beta}(P,T)}{\partial T^2}(dT)^2 = 0
$$

where,

$$
\begin{aligned}
\frac{\partial^2 G_{\alpha\beta}(P,T)}{\partial P^2} &= \frac{\partial V}{\partial P}\Big|_T \\
\frac{\partial^2 G_{\alpha\beta}}{\partial P\partial T} &= \frac{\partial V}{\partial P}\Big|_P \\
\frac{\partial^2 G_{\alpha\beta}(P,T)}{\partial T^2} &= -\frac{C_p}{T}
\end{aligned}
$$

Chapter 2

Scope and Objective

2.1 Scope

Black hole is always known for the study of quantum gravity.On the other hand, it can be considered as a thermodynamic system with unique properties.The current understanding of gravity is in terms of geometry of space and time.Quantum gravity is the study of gravity using quantum mechanical principles.AdS/CFT correspondence is one of the correlation that deals with quantum gravity.

2.2 Objective

- To understand the behaviour of AdS space.
- To understand the causal structure of AdS space
- To understand the causal structure of RN-AdS black hole
- To study the thermodynamical behaviour of RN-AdS black hole
- To study the thermodynamical behaviour of Bardeen black hole and to understand the phase transition below T_c

Chapter 3

Conformal Diagrams

Conformal diagram is a two-dimensional space-time diagram which always has a finite size. Using the geodesic equation, the light geodesics are always represented by straight lines at $\pm$ $45°$. Finite ranges obtained by manipulation can be altered to rescale the diagram but its shape has to unique. [Muk15]

3.1 Minkowski space-time

The metric of Minkowski space-time in polar coordinates is given by,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of 2-sphere.
The range be $-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Since θ and ϕ are already finite. There is no need to rescale θ and ϕ coordinate. Hence it is enough if we rescale time and radial coordinate to draw a diagram in finite range.
Initially, we switch to null coordinates from original coordinates to rescale the range. Introducing the null coordinates u and v as u = t + r v = t - r.Thus t and r can be rewritten as

$$t = \frac{1}{2}(u + v) \qquad\qquad r = \frac{1}{2}(u - v)$$

Since r $\geq$ 0, u $\geq$ v. Now the metric be

$$ds^2 = -\frac{dudv + dvdu}{2} + \frac{(u - v)^2}{4} d\Omega^2$$

Now to bring the range to a finite coordinate value, choosing

$$U = \arctan(u) \qquad\qquad V = \arctan(v)$$

with ranges,$-\frac{\pi}{2} < U < \frac{\pi}{2}, \quad -\frac{\pi}{2} < V < \frac{\pi}{2}$, U $\geq$ V. So that the metric in new coordinates be,

$$ds^2 = -\frac{\sec^2 U \sec^2 V (dUdV + dVdU)}{2} + \frac{(\tan U - \tan V)^2}{4} d\Omega^2$$

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\Big(- 2(dUdV + dVdU) + \sin^2(U - V) d\Omega^2\Big)$$

Just to make the metric simpler, we introduce T and R coordinate as $T = U + V, \quad R = U - V$.Then U and V become,

$$U = \frac{1}{2}(T + R) \qquad\qquad V = \frac{1}{2}(T - R)$$

with ranges $-\pi < T + R < \pi, \quad -\pi < T - R < \pi$. The metric is,

$$ds^2 = \Omega^{-2}(T, R)(-dT^2 + dR^2 + \sin^2 R d\Omega^2)$$

where Ω(T,R) $= \cos T + \cos R$, called the conformal factor.

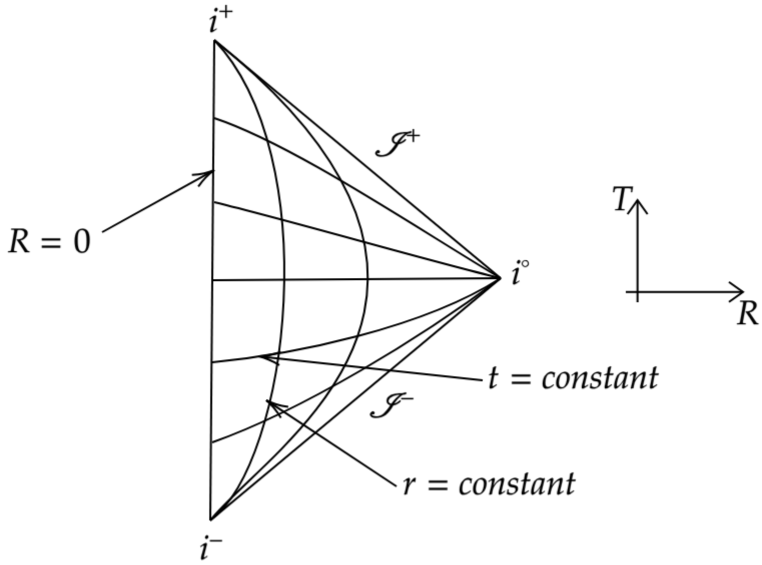

Figure 3.1: Conformal Diagram of Minkowski spacetime

where i° is called space-like infinity, i^- is past time-like infinity ,i^+ is future time-like infinity, past null infinity is denoted by $\mathcal{I}^-$ and future null infinity is denoted by $\mathcal{I}^+$.Each point in the conformal diagram represents 2-sphere.

3.2 Anti-de Sitter Space

To obtain the conformal diagram, ρ coordinate is transformed to χ coordinate system as

$$\cosh\rho = \frac{1}{\cos\chi} \quad thus, \quad \sinh\rho = \tan\chi$$

$$and \quad d\rho = \frac{1}{\cos\chi} d\chi$$

The metric changes to

$$ds^2 = \frac{l^2}{\cos^2\chi}\big(- dt^2 + d\chi^2 + \sin^2\chi d\Omega_2^2\big)$$

Here the range of time coordinate is from $-\infty$ to $+\infty$ and the range of radial coordinate turns out to be finite i.e., $-\frac{\pi}{2} \leq \chi \leq \frac{\pi}{2}$

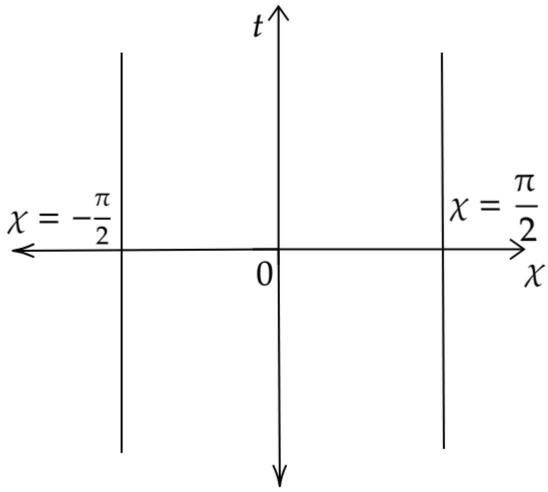

Figure 3.2: Null coordinates in AdS space

When we consider time coordinate is periodic, the conformal diagram changes to

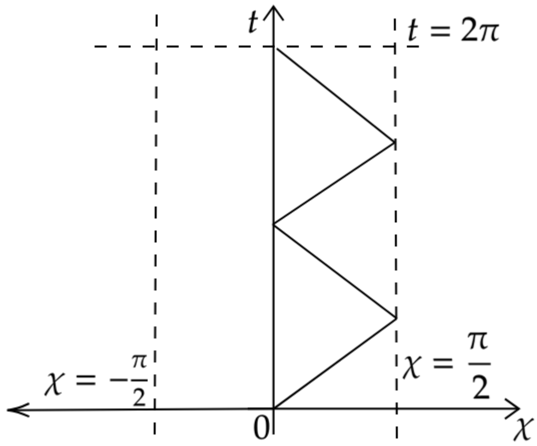

Figure 3.3: Conformal Diagram of Anti-de Sitter Spacetime when time is periodic

Lorentzian part is not enough to explain AdS space is maximally symmetric. It talks only about translational invariance. But the space has to be both translational as well as rotational invariance to show that the space is maximally symmetric. Hence we change the coordinates to Poincare coordinates. To avoid closed time like curve,one can take the range of time as $-\infty < t < \infty$ and can draw conformal diagram in a universal covering space. Poincare coordinates(t,x,y,z) are taken as[Sok16],

$$X_0 = \frac{1}{2z}(l^2 + x^2 + y^2 + z^2 + t^2)$$
$$X_1 = \frac{1}{2z}(l^2 - x^2 - y^2 - z^2 + t^2)$$

$$X_2 = l\frac{t}{z}$$
$$X_3 = l\frac{x}{z}$$

$$X_4 = l\frac{y}{z}$$

And these equations satisfy equation(1.1).Now to obtain the metric,

$$dX_0 = \frac{2z(xdx + ydy + zdz + tdt) - (l^2 + x^2 + y^2 + z^2 + t^2)2}{2z^2}$$

$$dX_1 = \frac{2z(-xdx - ydy - zdz + tdt) - (l^2 - x^2 - y^2 - z^2 + t^2)}{2z^2}$$

$$dX_2 = l\frac{zdt - tdz}{z^2}$$

$$dX_3 = l\frac{zdx - xdz}{z^2}$$

$$dX_4 = l\frac{zdy - ydz}{z^2}$$

Substituting in (1.2) and rearranging,

$$ds^2 = \frac{l^2}{z^2}(dt^2 - dx^2 - dy^2 - dz^2) \tag{3.1}$$

Here $z > 0$, this implies that the Poincare coordinates covers only the half of the AdS space.Then the conformal diagram be

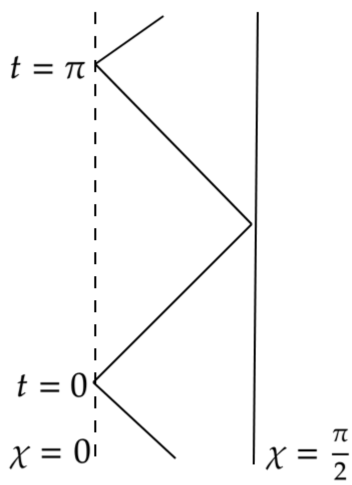

Figure 3.4: Conformal Diagram of AdS Space using Poincare Coordinates

Chapter 4

Reissner-Nordst$\ddot{o}$rm Black hole

4.1 Reissner-Nordst$\ddot{o}$rm Metric

The general metric of the charged black hole is[Car14]

$$ds^2 = -\exp^{2\alpha(r,t)} dt^2 + \exp^{2\beta(r,t)} dr^2 + r^2 d\Omega^2$$

The Reissner-Nordst$\ddot{o}$rm metric be[Car14]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of 2-sphere and

$$f(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2}$$

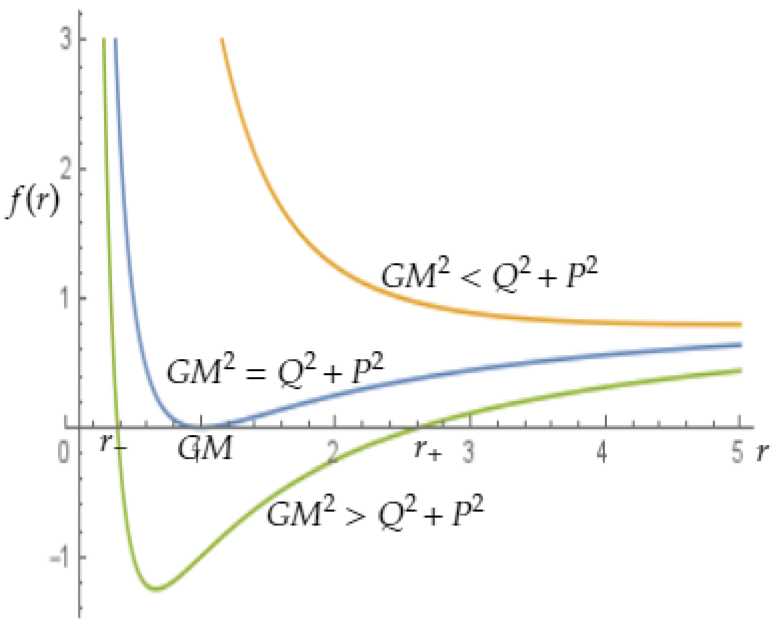

Figure 4.1: The variation of f(r) with r for Reissner-Nordstr$\ddot{o}$m solution

Along the event horizon, the function has the largest root. Hence it is equal to zero.

$$f(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} = 0$$

$$r^2 - 2GMr + G(Q^2 + P^2) = 0$$
$$r_{\pm} = GM \pm \sqrt{GM^2 - G(Q^2 + P^2)}$$

Then f(r) can be written in terms of r_+ and r_-.

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2} \tag{4.1}$$

4.2 Conformal Diagram of RN spacetime

Considering radial null curves for constant θ and ϕ,the geodesic equation be

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} = 0$$

$$\frac{dt^2}{dr^2} = \frac{1}{f^2(r)}$$

Then the slope of the light cone on t-r plane is,

$$\frac{dt}{dr} = \pm\frac{1}{f(r)}$$

$$\frac{dt}{dr} = \pm\frac{1}{1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}}$$

For large r, the slope $\frac{dt}{dr} = \pm 1$

Thus as r approaches infinity,the solution approaches flat spacetime.Then the metric is as same as that of Minkowski metric.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

Thus the conformal diagram is same as that of Minkowski space except at r=0 where the singularity exists.

case 1:$GM^2 = Q^2 + P^2$

This is the extreme case of Reissner-Nordstörm solution. Here, $r_{\pm} = GM$, implies that the inner horizon and the outer horizon coincide with each other. The total energy of black hole is equal to the energy of the electric field. When r = GM, the slope $\frac{dt}{dr} = \pm\infty$.Thus the light ray approaching r = GM looks like it is reaching r asymptotically.

case 2:$GM^2 < Q^2 + P^2$

For the condition $GM^2 < Q^2 + P^2$, the function f(r) is positive and r is complex function, thus the event horizon doesn't exist. But r=0 implies the geodesic is purely timelike. Since there is no obstruction to a observer to go to the singularity and get the instructions. The singularity is called *naked singularity.*

But in this case, no where we can expect a black hole in the space. Because, $GM^2 < Q^2 + P^2$ represents the gravitational collapse of the black hole since the total energy of the black hole is less than the energy of the electric field.

case 3:$GM^2 > Q^2 + P^2$

This is the standard Reissner-Nordstörm solution where the total mass of the black hole is greater than energy of an electric field. Here $r_{\pm} = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2)}$. Thus there exists two horizons, inner horizon r_- and the outer horizon r_+.From the graph (4.1) it is clear that the function f(r) is positive at large r and small r, but negative in between r_+ and r_-. Using equation (4.1), the metric can be written as,

$$ds^2 = -\frac{1}{r^2}(r-r_+)(r-r_-)dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega^2$$

Now we transform the radial coordinate such a way that the metric is conformally flat.

$$dR = \frac{r}{\sqrt{(r-r_+)(r-r_-)}}dr$$

$$R = \int \frac{r}{\sqrt{(r-r_+)(r-r_-)}}dr$$

$$R(r) = r + \frac{1}{(r_+ - r_-)}(r_+^2 \ln(r-r_+) - r_-^2 \ln(r-r_-))$$

when r=0(singularity),

$$R(0) = \frac{1}{(r_+ - r_-)}(r_+^2 \ln(r_+) - r_-^2 \ln(r_-))$$

Since $d\Omega^2$ doesn't effect the global causal structure,we can consider only the Lorentzian part of the metric.

$$ds^2 = \frac{1}{r^2}(r-r_+)(r-r_-)(-dt^2 + dR^2)$$

Changing the coordinates t,R to null coordinate system as,

$$t = \frac{1}{2}\frac{\sin T}{\cos\frac{(T-R')}{2}\cos\frac{(T+R')}{2}}$$

$$R = \frac{1}{2}\frac{\sin R'}{\cos\frac{(T-R')}{2}\cos\frac{(T+R')}{2}}$$

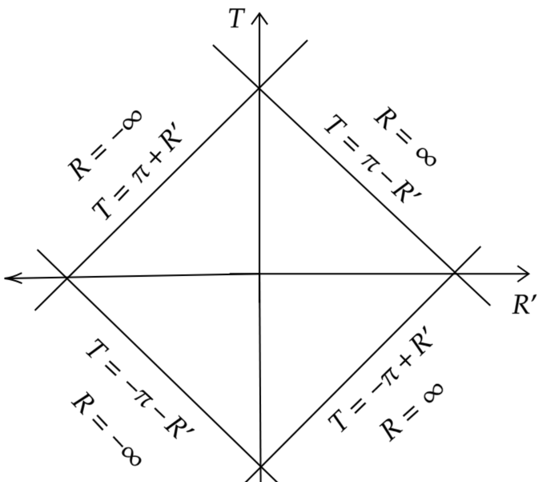

Figure 4.2: Null Coordinates of RN spacetime

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2 \cos^2\frac{(T+R')}{2} \cos^2\frac{(T-R')}{2}}(-dT^2 + dR'^2)$$

Since at r = $r_\pm$,coordinates break down,we divide the range of radial coordinate into the union of three ranges.

In the first range $0 < r < r_-$,

$$\lim_{r \to r_-} R(r) = +\infty$$

In the range $r_- < r < r_+$,

$$\lim_{r \to r_+} R(r) = -\infty \qquad \lim_{r \to r_-} R(r) = +\infty$$

In the range $r > r_+$,

$$\lim_{r \to \infty} R(r) = \infty \qquad \lim_{r \to r_+} R(r) = -\infty$$

Transforming $R' \to -R'$ and $T \to -T$ and collect the limiting values as earlier. Then joining all patches together gives the conformal diagram.

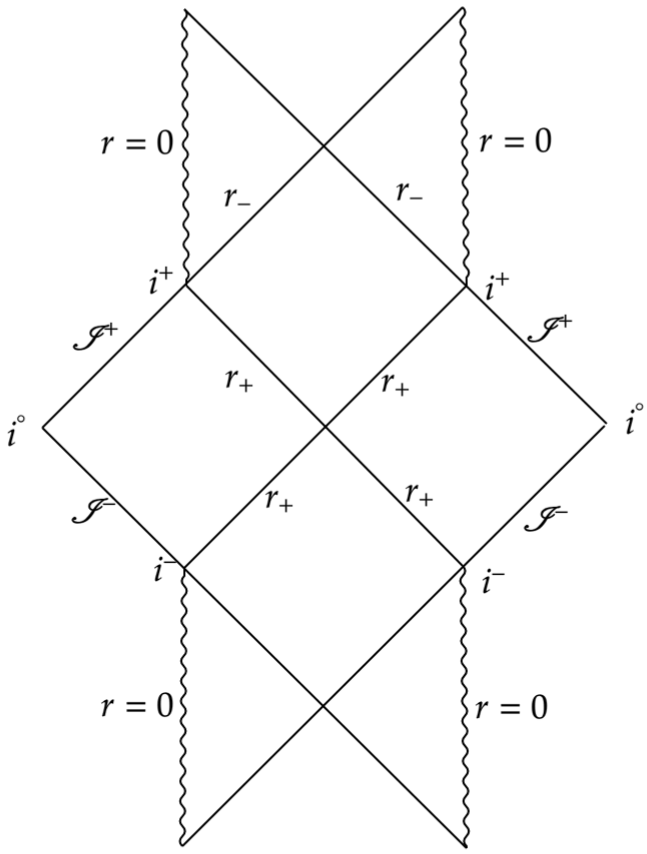

Figure 4.3: Conformal Diagram of RN spacetime when $GM^2 > G(Q^2 + P^2)$

Chapter 5

Thermodynamics of RN-AdS Black hole

The behaviour of any system can be understood by studying the thermodynamics of that particular system.And statistical mechanics has always been a tool to understand the thermodynamics. The existence of event horizon of a black hole is the basis for the study of thermodynamics of black hole. General laws of thermodynamics is analogous to the black hole thermodynamics.In this chapter, we are going to study the behaviour RN-AdS black hole which mathematically relates to Vander Waals fluid.

5.1 Black hole Thermodynamics

The first law of thermodynamics can be written as

$$dE = TdS - PdV + work\ terms$$

By our usual understanding, we know Energy E is directly proportional to Mass M.The zeroth law of black hole thermodynamics states that the surface gravity κ of a stationary black hole is constant throughout its event horizon.The law can be compared with the zeroth law of thermodynamics which states that the temperature T of the system remains constant for a system in thermal equilibrium.Thus the relationship between temperature and surface gravity can be written as, $T = \frac{\kappa}{2\pi}$

Hawking's area theorem says that the area of the event horizon cannot be decreased.This theorem is compared with the second law of thermodynamics which says that the entropy of the black hole can never decrease.Thus the Entropy and area of the event horizon have the monotonic relation as $\frac{A}{4}$, where A is the area of the event horizon.

Thus the first law of black hole thermodynamics can be written as,

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

Here, ΩdJ and ΦdQ are work terms.

The first law of thermodynamics is incompatible with the Smarr relation unless variable cosmological constant Λ is included in the first law.Comparing first law of thermodynamics with the black hole thermodynamic equation under variable Λ,Mass is identified as enthalpy rather than internal energy.[KM12]

$$M = E + PV$$

For $\Lambda < 0$, cosmological constant in a D-dimensional AdS space with radius l is given by the equation[KM14]

$$\Lambda = -\frac{(D-1)(D-2)}{l^2}$$

Since Λ corresponds to pressure, the pressure of asymptotic AdS black hole is always positive. The pressure of the AdS black in 4-dimension is given by[KM12]

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}$$

5.2 Equation of State

Now including the variable Λ term and taking magnetic charge P=0, the RN metric can be rewritten as[KM12]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d{\Omega_2}^2 \tag{5.1}$$

$$where \;\; f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}$$

Along the event horizon, the function has the largest root. Thus, $f(r_+)$ is zero.i.e.,

$$1 - \frac{2M}{r_+} + \frac{Q^2}{{r_+}^2} + \frac{{r_+}^2}{l^2} = 0$$

On rearrangement,

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} + \frac{{r_+}^3}{2l^2} \tag{5.2}$$

where r_+ is the radius of the event horizon.
From the first law of thermodynamics, we can write an equation for temperature as,

$$T = \Big(\frac{\partial M}{\partial S}\Big)_{V\cdots}$$

Differentiating equation (5.2) with respect to r_+,

$$\frac{dM}{dr_+} = \frac{1}{2} - \frac{Q^2}{2r_+^2} + \frac{3r_+^2}{2l^2}$$

$$\frac{dM}{dr_+} = \frac{1}{2}\Big[1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}\Big] \tag{5.3}$$

We know S = $\frac{A}{4}$ where A = 4 πr_+^2. Differentiating with respect to r_+,

$$\frac{dS}{dr_+} = \frac{1}{4}\frac{dA}{dr_+} = 2\pi r_+ \tag{5.4}$$

Dividing (5.3) by (5.4),

$$T = \frac{\frac{dM}{dr_+}}{\frac{dS}{dr_+}} = \frac{\frac{1}{2}\Big[1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}\Big]}{2\pi r_+}$$

$$T = \frac{1}{4\pi r_+}\Big[1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}\Big] \tag{5.5}$$

Divide the above equation by $2r_+$ throughout,

$$\frac{T}{2r_+} = \frac{1}{8\pi r_+^2} - \frac{Q^2}{8\pi r_+^4} + \frac{3}{8\pi l^2} \tag{5.6}$$

Introducing pressure term $P = \frac{3}{8\pi l^2}$ in the above equation,

$$\frac{T}{2r_+} = \frac{1}{8\pi r_+^2} - \frac{Q^2}{8\pi r_+^4} + P$$

On rearrangement,

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} \tag{5.7}$$

Equation (5.7) is the **equation of state** for a charged AdS black hole.
Before starting further calculation, we have to translate this geometric equation into a physical one. The physical pressure and physical temperature is given by,

$$Pressure = \frac{\hbar c}{l_P^2}P \qquad\qquad Temperature = \frac{\hbar c}{k}T$$

where Planck length $l_P = \sqrt{\frac{\hbar G}{c^3}}$. Multiplying the both side of the equation (5.7) by $\frac{\hbar c}{l_P^2}$,

$$Pressure = \frac{\hbar c}{l_P^2}P = \frac{\hbar c}{l_P^2}\frac{T}{2r_+} - \frac{\hbar c}{l_P^2}\frac{1}{8\pi r_+^2} + \frac{\hbar c}{l_P^2}\frac{Q^2}{8\pi r_+^4}$$

$$Pressure = -\frac{k}{2l_P^2 r_+}Temperature - \frac{\hbar c l_P^2}{2\pi}\frac{1}{4l_P^4 r_+^2} + \frac{Q^2\hbar c l_P^6}{\pi}\frac{2}{16 l_P^8 r_+^4} \tag{5.8}$$

Comparing this with the Vander waals equation,

$$P = \frac{kT}{v-b} - \frac{a}{v^2}$$

where $v = V/N$. Hence we can conclude that the specific volume $v = 2l_P^2 r_+ = 2l_P^2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$ with N $= \frac{1}{l_P^2}$. Returning back to the geometrical units, equation (5.8) can be rewritten as,

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4} \tag{5.9}$$

Thus gives the **equation of state**.

5.3 Law of Corresponding States

At inflection point,

$$\frac{\partial P}{\partial v} = 0, \qquad \frac{\partial^2 P}{\partial v^2} = 0$$

$$\left(\frac{\partial P}{\partial v}\right)_{v=v_c} = -\frac{T_c}{v_c^2} + \frac{2}{2\pi v_c^3} - \frac{8Q^2}{\pi v_c^5} = 0$$

$$\frac{1}{\pi v_c} - \frac{8Q^2}{\pi v_c^3} = T_c \tag{5.10}$$

$$\left(\frac{\partial^2 P}{\partial v^2}\right)_{v=v_c} = \frac{2T_c}{v_c^3} - \frac{3}{\pi v_c^4} - \frac{40Q^2}{\pi v_c^6} = 0$$

$$\frac{3}{\pi v_c} - \frac{40Q^2}{\pi v_c^3} = 2T_c \tag{5.11}$$

Dividing (5.10) by (5.11) gives,

$$\frac{\frac{1}{\pi v_c} - \frac{8Q^2}{\pi v_c^3}}{\frac{3}{\pi v_c} - \frac{40Q^2}{\pi v_c^3}} = \frac{T_c}{2T_c}$$

$$\frac{v_c^2 - 8Q^2}{3v_c^2 - 40Q^2} = \frac{1}{2}$$

$$2v_c^2 - 16Q^2 = 3v_c^2 - 40Q^2$$

$$v_c^2 = 24Q^2$$

$$v_c = 2\sqrt{6}Q \tag{5.12}$$

Rearranging equation (5.10) and substituting the value of v_c,

$$\begin{aligned} T_c &= \frac{v_c - 8Q^2}{\pi v_c^3} \\ &= \frac{24Q^2 - 8Q^2}{48\sqrt{6}\pi Q^3} \\ &= \frac{16Q^2}{48\sqrt{6}\pi Q^3} \\ &= \frac{1}{3\sqrt{6}\pi Q} \end{aligned}$$

$$T_c = \frac{\sqrt{6}}{18\pi Q} \tag{5.13}$$

Substituting the value of v_c and T_c in equation (5.9),

$$\begin{aligned} P_c &= \frac{\sqrt{6}}{18\pi Q}\frac{1}{2\sqrt{6}Q} - \frac{1}{2\pi(2\sqrt{6}Q)^2} + \frac{2Q^2}{\pi(2\sqrt{6}Q)^4} \\ &= \frac{1}{36\pi Q^2} - \frac{1}{48\pi Q^2} + \frac{1}{288\pi Q^2} \end{aligned}$$

$$P_c = \frac{1}{96\pi Q^2} \tag{5.14}$$

Thus we can write an interesting relation,

$$\frac{P_c v_c}{T_c} = \frac{\frac{1}{96\pi Q^2}2\sqrt{6}Q}{\frac{\sqrt{6}}{18\pi Q}}$$

$$\frac{P_c v_c}{T_c} = \frac{3}{8} \tag{5.15}$$

which is same as that of Vander Waals fluid.Here P_c, v_c and T_c are respectively known as critical pressure, critical volume and critical temperature.

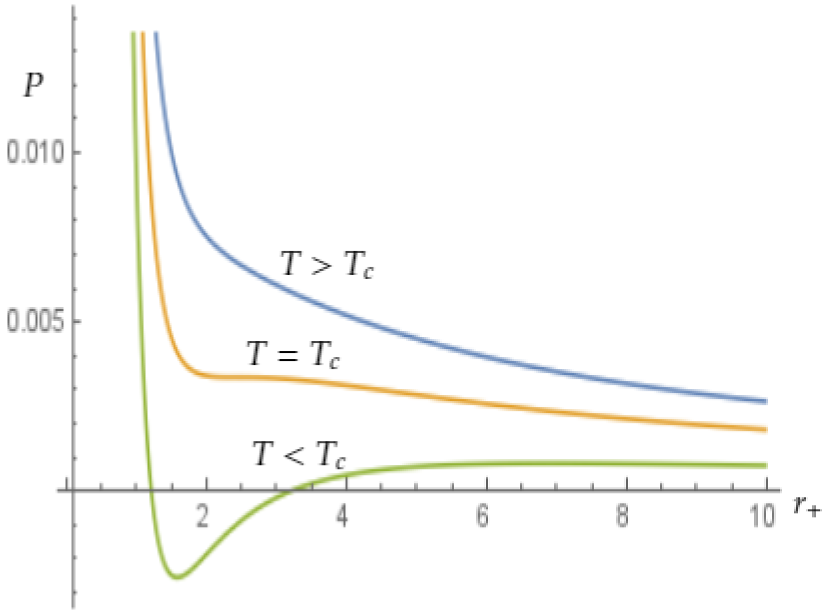

Figure 5.1: P-V Diagram of RN-AdS Black hole when Q=1

Now reduced pressure, reduced volume and reduced temperature can be written as

$$T' = \frac{T}{T_c} \qquad P' = \frac{P}{P_c} \qquad v\prime = \frac{v}{v_c}$$

Then P ,$v\prime$ and T can be written as,

$$P = \frac{P'}{96\pi Q^2} \qquad v = 2\sqrt{6}Qv\prime \qquad T = \frac{\sqrt{6}T'}{18\pi Q} \tag{5.16}$$

Substituting equation (5.16) in (5.9),

$$\frac{P'}{96\pi Q^2} = \frac{\sqrt{6T'}}{18\pi Q}\frac{1}{2\sqrt{6}Qv\prime} - \frac{1}{2\pi(2\sqrt{6}Qv\prime)^2} + \frac{2Q^2}{\pi(2\sqrt{6}Qv\prime)^4}$$

$$\frac{P'}{96\pi Q^2} = \frac{T'}{36\pi Q^2 v\prime} - \frac{1}{48\pi Q^2 v\prime^2} + \frac{1}{288\pi Q^2 v\prime^4}$$

$$\frac{P'}{8} = \frac{T'}{3v\prime} - \frac{1}{4v\prime^2} + \frac{1}{24v\prime^4}$$

$$\frac{1}{8}\left[P' + \frac{2}{v\prime^2} - \frac{1}{3v\prime^4}\right] = \frac{T'}{3v\prime}$$

$$3v\prime\left[P' + \frac{2}{v\prime^2}\right] - \frac{1}{v\prime^3} = 8T' \tag{5.17}$$

This is called **Law of corresponding states**.

Chapter 6

Bardeen Black Hole

Black holes are one of the most mysterious and fascinating objects predicted theoretically in General Relativity(GR) which has become reality because of recent real black hole image. GR predicts there exists singularity at the centre of black hole neglecting quantum mechanical effects. But these singularities can be avoided depending on where singularity is present. If there exist coordinate singularity, it can be removed by transforming the coordinates, if there exist curvature singularity, which usually be present at the centre of the system and that cannot be removed. And these singularity-free black holes are known as regular black holes. One of the first regular black hole model is proposed by Bardeen in 1968.[Tzi19]

6.1 Equations of Motion

The corresponding action for a given cosmological constant Λ is given by[Car14]

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G}(R - 2\Lambda) + \hat{\mathcal{L}}_M\Big]$$

where g is the determinant of the metric tensor, R is the Ricci scalar, $\hat{\mathcal{L}}_M$ is a matter Lagrangian. For $\Lambda < 0$, cosmological constant in a D-dimensional AdS space with radius l is given by the equation[KM14]

$$\Lambda = -\frac{(D-1)(D-2)}{l^2}$$

The action turns out to be

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(R + \frac{6}{l^2} - 4\mathcal{L}(F)\Big) \tag{6.1}$$

where $\mathcal{L}(F)$ is a function of F given by[Ayó00],

$$\mathcal{L}(F) = \frac{3M}{|q|^3}\Big(\frac{\sqrt{4q^2F}}{1+\sqrt{4q^2F}}\Big)^{\frac{5}{2}} \tag{6.2}$$

Varying the action with respect to inverse metric,

$$
\begin{aligned}
\frac{\delta S}{\delta g^{\mu\nu}} &= \frac{1}{16\pi G}\int d^4x \frac{\delta}{\delta g^{\mu\nu}}\Big[\sqrt{-g}\Big(R+\frac{6}{l^2}-4\mathcal{L}(F)\Big)\Big] \\
&= \frac{1}{16\pi G}\int d^4x \Big[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\Big(R+\frac{6}{l^2}-4\mathcal{L}(F)\Big)+\sqrt{-g}\Big(\frac{\delta R}{\delta g^{\mu\nu}}+\frac{\delta}{\delta g^{\mu\nu}}\Big(\frac{6}{l^2}-4\frac{\delta\mathcal{L}(F)}{\delta g^{\mu\nu}}\Big)\Big)\Big] \\
&= \frac{1}{16\pi G}\int d^4x \Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}R-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\frac{6}{l^2}+2\sqrt{-g}g_{\mu\nu}\mathcal{L}(F)+\sqrt{-g}\Big(\frac{\delta R}{\delta g^{\mu\nu}}-4\frac{\delta\mathcal{L}(F)}{\delta g^{\mu\nu}}\Big)\Big]
\end{aligned}
\tag{6.3}
$$

where $\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}$ = - $\frac{1}{2}\sqrt{-g}\ g_{\mu\nu}$ Ricci scalar is wriiten as

$$R = g^{\sigma\rho}R_{\sigma\rho}$$

Varying the Ricci scalar with respect to the metric tensor,

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\sigma\rho}\frac{\delta g^{\sigma\rho}}{\delta g^{\mu\nu}} + g^{\sigma\rho}\frac{\delta R_{\sigma\rho}}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{6.4}$$

Varying the Lagrangian with respect to the metric tensor,

$$\frac{\delta\mathcal{L}(F)}{\delta g^{\mu\nu}} = \frac{\delta\mathcal{L}(F)}{\delta F}\frac{\delta F}{\delta g^{\mu\nu}}$$

Field tensor can also be written as

$$F = F^{\sigma\rho}F_{\sigma\rho} = g^{\sigma\alpha}g^{\rho\beta}F_{\alpha\beta}F_{\sigma\rho}$$

Varying the field tensor with respect to the metric tensor,

$$\frac{\delta F}{\delta g^{\mu\nu}} = F_{\alpha\beta}F_{\sigma\rho}\Big[\frac{\delta g^{\sigma\alpha}}{\delta g^{\mu\nu}}g^{\rho\beta} + g^{\sigma\alpha}\frac{\delta g^{\rho\beta}}{\delta g^{\mu\nu}}\Big] = 2F_{\mu\lambda}F^{\lambda}_{\nu} \tag{6.5}$$

Substituting (6.4) and (6.5) in (6.3)

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}R-\frac{3}{l^2}g_{\mu\nu}+2g_{\mu\nu}\mathcal{L}(F)+R_{\mu\nu}-2\frac{\delta\mathcal{L}(F)}{\delta F}2F_{\mu\lambda}F^{\lambda}_{\nu}\Big]$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}R-\frac{3}{l^2}g_{\mu\nu}+2g_{\mu\nu}\mathcal{L}(F)+R_{\mu\nu}-2\frac{\delta\mathcal{L}(F)}{\delta F}2F_{\mu\lambda}F^{\lambda}_{\nu} = 0$$

$$G_{\mu\nu}-\frac{3}{l^2}g_{\mu\nu}+2g_{\mu\nu}\mathcal{L}(F)-2\frac{\delta\mathcal{L}(F)}{\delta F}2F_{\mu\lambda}F^{\lambda}_{\nu} = 0$$

$$\boxed{G_{\mu\nu}-\frac{3}{l^2}g_{\mu\nu} = 2\Big(\frac{\delta\mathcal{L}(F)}{\delta F}2F_{\mu\lambda}F^{\lambda}_{\nu}-g_{\mu\nu}\mathcal{L}(F)\Big)} \tag{6.6}$$

Now varying the action with respect to scalar potential,

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\nu} = \frac{\partial\mathcal{L}}{\partial A_\nu} - \nabla_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu A_\nu)}\Big) = 0$$

$$where \quad \frac{\partial\mathcal{L}}{\partial(\nabla_\mu A_\nu)} = \frac{\partial\mathcal{L}}{\partial F}\frac{\partial F}{\partial(\nabla_\mu A_\nu)}$$

Field tensor becomes

$$F = F^{\sigma\rho}F_{\sigma\rho} = g^{\sigma\alpha}g^{\rho\beta}F_{\alpha\beta}F_{\sigma\rho}$$

Varying the field tensor with respect to scalar potential,

$$\frac{\partial F}{\partial(\nabla_\mu A_\nu)} = g^{\sigma\alpha}g^{\rho\beta}\Big[\frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)}F_{\sigma\rho} + F_{\alpha\beta}\frac{\partial F_{\sigma\rho}}{\partial(\nabla_\mu A_\nu)}\Big]$$

$$\begin{aligned} where \quad F_{\alpha\beta} &= 2(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha) \\ F_{\sigma\rho} &= 2(\nabla_\sigma A_\rho - \nabla_\rho A_\sigma) \end{aligned}$$

$$\frac{\partial F}{\partial(\nabla_\mu A_\nu)} = 8F^{\mu\nu}$$

$$and \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Thus the equation of motion becomes

$$\boxed{\nabla_\mu\Big(\frac{\partial \mathcal{L}}{\partial F}\ F^{\mu\nu}\Big) = 0} \tag{6.7}$$

Considering the ansatz made in [Ayó00] for the Maxwell field,

$$F_{\mu\nu} = 2\delta^\theta_{[\mu}\delta^\phi_{\nu]}F(\theta,\phi)$$

Substituting the above equation in (6.7) and integrating[Fer16],

$$F_{\mu\nu} = 2\delta^\theta_{[\mu}\delta^\phi_{\nu]}q(r)\sin\theta$$

Equation (6.7) implies

$$dF = \frac{dq}{dr}\sin\theta dr \wedge d\theta \wedge d\phi = 0$$

This leads to the conclusion that q(r) =constant=q.Thus gives[Ayó00]

$$F_{\theta\phi} = 2q\sin\theta \qquad\qquad F = \frac{q^2}{2r^4}$$

Substituting the value of F in (6.2), we get

$$\mathcal{L}(F) = \mathcal{L}(r) = \frac{3Mq^2}{(r^2+q^2)^{5/2}} \tag{6.8}$$

The given metric be

$$ds^2 = -\Big(1-\frac{2m(r)}{r}\Big)dt^2 + \frac{dr^2}{\Big(1-\frac{2m(r)}{r}\Big)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.9}$$

$$g_{\mu\nu} = \begin{bmatrix} -\Big(1-\frac{2m(r)}{r}\Big) & 0 & 0 & 0 \\ 0 & \frac{1}{\Big(1-\frac{2m(r)}{r}\Big)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\left(1-\frac{2m(r)}{r}\right)} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m(r)}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix}$$

To calculate the Einstein tensor, we begin with the non-vanishing christoffel symbols which are given by,

$$\begin{aligned}
\Gamma^{t}_{tr} &= -\frac{1}{1-\frac{2m(r)}{r}}\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
\Gamma^{\theta}_{r\theta} &= \frac{1}{r} \\
\Gamma^{r}_{\phi\phi} &= -\Big(1-\frac{2m(r)}{r}\Big) r\sin^2\theta \\
\Gamma^{r}_{tt} &= -\Big(1-\frac{2m(r)}{r}\Big)\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
\Gamma^{r}_{\theta\theta} &= -r\Big(1-\frac{2m(r)}{r}\Big) \\
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{r}_{rr} &= \frac{1}{1-\frac{2m(r)}{r}}\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{r} \\
\Gamma^{\phi}_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The non-vanishing component of Reimann tensor:

$$\begin{aligned}
R^{t}_{rtr} &= \frac{1}{1-\frac{2m(r)}{r}}\Big(\frac{\partial_r^2 m(r)}{r} - 2\frac{\partial_r m(r)}{r^2} + 2\frac{m(r)}{r^3}\Big) \\
R^{t}_{\theta t\theta} &= r\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
R^{t}_{\phi t\phi} &= r\sin^2\theta\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
R^{r}_{\theta r\theta} &= r\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
R^{r}_{\phi r\phi} &= r\sin^2\theta\Big(\frac{\partial_r m(r)}{r} - \frac{m(r)}{r^2}\Big) \\
R^{\theta}_{\phi\theta\phi} &= 2\sin^2\theta\frac{m(r)}{r}
\end{aligned}$$

Taking the contraction yields the Ricci tensor:

$$\begin{aligned}
R_{tt} &= -\Big(1-\frac{2m(r)}{r}\Big)\Big(\frac{\partial_r^2 m(r)}{r}\Big) \\
R_{rr} &= \frac{1}{\Big(1-\frac{2m(r)}{r}\Big)}\Big(\frac{\partial_r^2 m(r)}{r}\Big) \\
R_{\theta\theta} &= 2\ \partial_r m(r) \\
R_{\phi\phi} &= 2\ \sin^2\theta\ \partial_r m(r)
\end{aligned}$$

Thus gives the curvature scalar,

$$R^t_t = \Big(\frac{\partial_r^2 m(r)}{r}\Big) \qquad\qquad R^r_r = \Big(\frac{\partial_r^2 m(r)}{r}\Big)$$

$$R^\theta_\theta = \frac{2}{r^2}\,\partial_r m(r) \qquad\qquad R^\phi_\phi = \frac{2}{r^2}\,\partial_r m(r)$$

$$\begin{aligned} R &= R^t_t + R^r_r + R^\theta_\theta + R^\phi_\phi \\ &= 2\Big(\frac{\partial_r^2 m(r)}{r}\Big) + \frac{4}{r^2}\,\partial_r m(r) \end{aligned}$$

Thus the Einstein tensor be

$$\begin{aligned} G_{tt} &= R_{tt} - \frac{1}{2} R g_{tt} \\ &= -\frac{2}{r^2}\Big(1 - \frac{2m(r)}{r}\Big)\partial_r m(r) \end{aligned}$$

Rewritting the obtained equation of motion as

$$\begin{aligned} G_{tt} - \frac{3}{l^2} g_{tt} &= 2\Big(\frac{\delta \mathcal{L}(F)}{\delta F} 2F_{t\lambda}F^\lambda_t - g_{tt}\mathcal{L}(F)\Big) \\ &= -2 g_{tt}\mathcal{L}(F) \end{aligned} \tag{6.10}$$

Since $F_{t\lambda}F^\lambda_t = 0$.
Substituting the value for G_{tt}, g_{tt},

$$-\frac{2}{r^2}\Big(1 - \frac{2m(r)}{r}\Big)\partial_r m(r) - \frac{3}{l^2}\Big(1 - \frac{2m(r)}{r}\Big) = -2\Big(1 - \frac{2m(r)}{r}\Big)\frac{3Mq^2}{(r^2+q^2)^{\frac{5}{2}}} \tag{6.11}$$

$$\partial_r m(r) + \frac{3r^2}{2l^2} = \frac{3Mq^2 r^2}{(r^2+q^2)^{\frac{5}{2}}} \tag{6.12}$$

Integrating equation (6.12) from r to ∞,

$$m(r) + \frac{r^3}{2l^2} + Constant = \int_r^\infty \frac{3Mq^2r^2}{(r^2+q^2)^{\frac{5}{2}}} dr$$

$$\text{Constant} = \lim_{r\to\infty}\Big(m(r) + \frac{r^3}{2l^2}\Big) = M$$

Consider the integral, $\int_r^\infty \frac{3Mq^2r^2}{(r^2+q^2)^{\frac{5}{2}}} dr$, substituting r = q $\tan\theta$

$$\int_r^\infty \frac{3Mq^2r^2}{(r^2+q^2)^{\frac{5}{2}}} dr = 3M\int_{\arctan\frac{r}{q}}^{\frac{\pi}{2}} \sin^2\theta\cos\theta d\theta$$

Let $\sin\theta = t$,

$$\begin{aligned} 3M\int_{\arctan\frac{r}{q}}^{\frac{\pi}{2}} \sin^2\theta\cos\theta d\theta &= 3M\int_{\sin\arctan\frac{r}{q}}^{1} t^2 dt \\ &= M + \frac{Mr^3}{(r^2+q^2)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} m(r) + \frac{r^3}{2l^2} + M &= M + \frac{Mr^3}{(r^2+q^2)^{\frac{3}{2}}} \\ m(r) &= \frac{Mr^3}{(r^2+q^2)^{3/2}} - \frac{r^3}{2l^2} \end{aligned} \tag{6.13}$$

Substituting the m(r) equation in f(r) which is given by,

$$f(r) = 1 - \frac{2m(r)}{r}$$

$$\boxed{f(r) = 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}} + \frac{r^2}{l^2}} \tag{6.14}$$

Consider a degenerate black hole which satisfies

$$f(r) = 0 = \frac{\partial f(r)}{\partial r}$$

Considering l = 1,the above equation gives the solution

$$r_o = \frac{\sqrt{-1+\sqrt{1+24q^2}}}{\sqrt{6}}$$

Substituting this solution back in the equation gives

$$m_o = \frac{\left(q^2 + \frac{1}{6}(-1+\sqrt{1+24q^2})\right)^{5/2}}{2q^2 - \frac{1}{6}(-1+\sqrt{1+24q^2})}$$

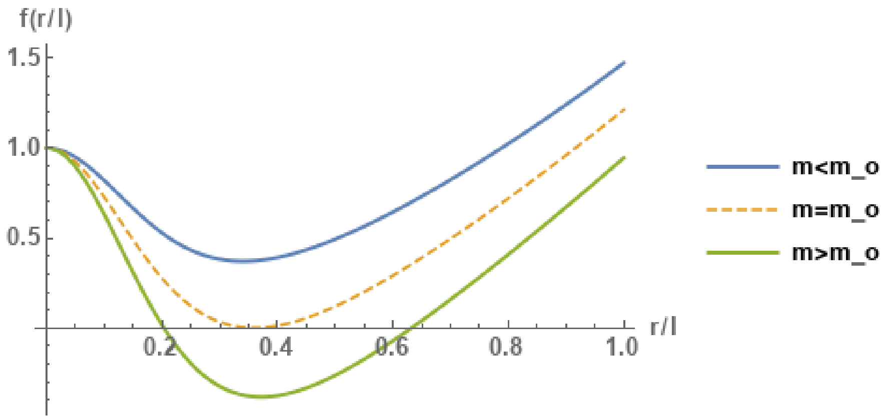

Figure 6.1: The variation of f(r) with respect to the radius of the black hole

- When m $<$ m$_o$, there is no horizon.
- When m $=$ m$_o$, there is degenerate horizon. This is the smallest possible radius and mass.
- When m $>$ m$_o$, there are two horizons i.e., inner horizon r$_-$ and outer horizon r$_+$.

Chapter 7

Thermodynamics of Bardeen Black hole

In order to study the thermal properties of BAdS black hole, we need to calculate all possible thermodynamic variables that describes BAdS black hole.
Along the event horizon, $f(r_+)$ is zero. Thus

$$1 - \frac{2Mr^2}{(r^2+q^2)^{\frac{3}{2}}} + \frac{r^2}{l^2} = 0$$

$$M = \Big(1 + \frac{r^2}{l^2}\Big)\frac{(r^2+q^2)^{\frac{3}{2}}}{2r^2}$$

$$\boxed{M = \frac{(l^2+r^2)(r^2+q^2)^{3/2}}{2l^2r^2}} \tag{7.1}$$

7.1 Temperature

The temperature of the stationary black hole can be calculated using the equation,

$$T = \frac{f'(r_+)}{4\pi}$$

Differentiating (6.14) with respect to r,

$$\begin{aligned} f'(r) &= -\frac{4Mr}{(q^2+r^2)^{3/2}} + \frac{6Mr^3}{(q^2+r^2)^{5/2}} + \frac{2r}{l^2} \\ &= \frac{2Mr}{(q^2+r^2)^{5/2}}(r^2-2q^2) + \frac{2r}{l^2} \end{aligned}$$

Substituting for M,

$$\begin{aligned} f'(r) &= \frac{2r(l^2+r^2)(r^2+q^2)^{3/2}(r^2-2q^2)}{2l^2r^2(q^2+r^2)^{5/2}} + \frac{2r}{l^2} \\ &= \frac{1}{l^2r}\Big[\frac{3r^4+l^2(r^2-2q^2)}{(q^2+r^2)}\Big] \end{aligned}$$

$$\boxed{T = \frac{3r_+^4+l^2(r_+^2-2q^2)}{4\pi l^2 r_+}(q^2+r_+^2)} \tag{7.2}$$

When T=T_c

$$\frac{\partial T}{\partial r_+} = 0 \qquad\qquad \frac{\partial^2 T}{\partial^2 r_+} = 0$$

where r_+ is the radius of the event horizon.

$$\frac{\partial T}{\partial r_+} = \frac{2r_+ + 12r_+^3}{4\pi r_+(q^2+r_+^2)} - \frac{-2q^2+r_+^2+3r_+^4}{2\pi(q^2+r_+^2)^2} - \frac{-2q^2+r_+^2+3r_+^4}{4\pi r_+^2(q^2+r_+^2)}$$

$$\frac{\partial^2 T}{\partial^2 r_+} = \frac{2+36r_+^2}{(4\pi r_+(q^2+r_+^2)} - \frac{2r_+ + 12r_+^3}{\pi(q^2+r_+^2)^2} - \frac{2x+12x^3}{2\pi r_+^2(q^2+r_+^2)} + \frac{2r_+(-2q^2+r_+^2+3r_+^4)}{\pi(q^2+r_+^2)^3}$$
$$+ \frac{(-2q^2+r_+^2+3r_+^4)}{2\pi r_+(q^2+r_+^2)^2} + \frac{(-2q^2+r_+^2+3r_+^4)}{2\pi r_+^3(q^2+r_+^2)}$$

The real root of the above equation be

$$q_c = \frac{\sqrt{219-13\sqrt{273}}}{12\sqrt{3}} l$$
$$\frac{q_c}{l} = Q_c \approx 0.0987$$

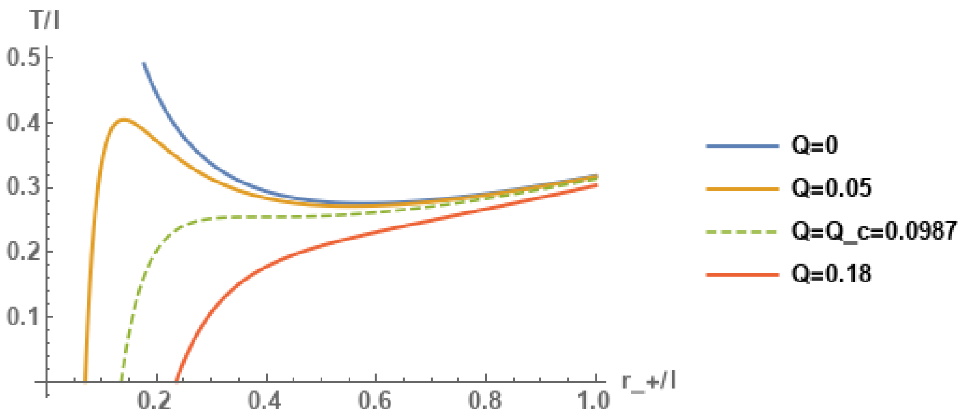

Figure 7.1: The variation of black hole temperature with respect to the radius of the black hole

In the above graph,

- Q < Q_c, there is one maximum and one minimum.
- Q = Q_c, there exist an inflection point where maximum and minimum merge.
- Q > Q_c, the temperature is constantly increasing with the radius.

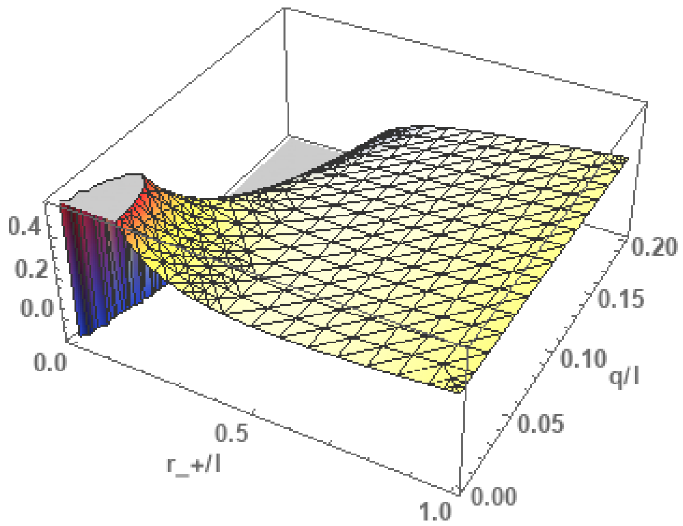

Figure 7.2: 3D temperature map of the black hole temprature by varying black hole radius and charge of the black hole simultaneously

From the above 3D plot, it is clear that Temperature is high when radius of the even horizon is small and charge is high.

7.2 Entropy

Using first law of thermodynamics,

$$T = \frac{\partial M}{\partial S} = \frac{\partial M/\partial r}{\partial S/\partial r}$$

$$\frac{\partial S}{\partial r} = \frac{1}{T}\frac{\partial M}{\partial r}$$

Differentiating (7.1) with respect to r,

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{(q^2+r^2)^{3/2}}{l^2 r} + \frac{3}{2}\frac{(l^2+r^2)(q^2+r^2)^{1/2}}{l^2 r} - \frac{(l^2+r^2)(q^2+r^2)^{3/2}}{l^2 r^3} \\ &= \frac{(q^2+r^2)^{1/2}}{2l^2r^3}\left[3r^4 + l^2(r^2-2q^2)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial r} &= \frac{1}{T}\frac{\partial M}{\partial r} \\ &= \frac{(q^2+r^2)^{1/2}}{2l^2r^3}\left[3r^4 + l^2(r^2-2q^2)\right]\frac{4\pi l^2 r(q^2+r^2)}{3r^4+l^2(r^2-2q^2)} \\ &= 2\pi\frac{(q^2+r^2)^{3/2}}{r^2} \\ &= 2\pi r\left(1+\frac{q^2}{r^2}\right)^{3/2}\end{aligned}$$

Integrating,

$$\begin{aligned}
S &= 2\pi \int_{r_o}^{r_+} r\Big(1+\frac{q^2}{r^2}\Big)^{3/2} dr \\
&= 2\pi \int_{r_o}^{r_+} r\Big(1+\frac{q^2}{r^2}\Big)\Big(1+\frac{q^2}{r^2}\Big)^{1/2} dr \\
&= 2\pi \int_{r_o}^{r_+} (r^2+q^2)^{1/2}\Big(1+\frac{q^2}{r^2}\Big)^{1/2} dr
\end{aligned}$$

Using Integration by parts,

$$\begin{aligned}
S &= 2\pi\Big[(r^2+q^2)^{1/2}\Big(r-\frac{q^2}{r}\Big)\Big|_{r_o}^{r_+} - \int_{r_o}^{r_+} \frac{r}{\sqrt{r^2+q^2}}\Big(r-\frac{q^2}{r}\Big)dr\Big] \\
&= 2\pi\Big[r\Big(1+\frac{q^2}{r^2}\Big)^{1/2} r\Big(1-\frac{q^2}{r^2}\Big)\Big|_{r_o}^{r_+} - \int_{r_o}^{r_+} \Big(\frac{r^2}{\sqrt{r^2+q^2}} - \frac{q^2}{\sqrt{r^2+q^2}}\Big)dr\Big] \\
&= 2\pi\Big[r^2\Big(1+\frac{q^2}{r^2}\Big)^{1/2}\Big(1-\frac{q^2}{r^2}\Big)\Big|_{r_o}^{r_+} - \int_{r_o}^{r_+} \Big(\frac{r^2+q^2-q^2}{\sqrt{r^2+q^2}} - \frac{q^2}{\sqrt{r^2+q^2}}\Big)dr\Big] \\
&= 2\pi\Big[r^2\Big(1+\frac{q^2}{r^2}\Big)^{1/2}\Big(1-\frac{q^2}{r^2}\Big)\Big|_{r_o}^{r_+} - \int_{r_o}^{r_+} \sqrt{r^2+q^2}dr + 2\int_{r_o}^{r_+} \frac{q^2}{\sqrt{r^2+q^2}}dr\Big] \\
&= 2\pi\Big[r^2\Big(1+\frac{q^2}{r^2}\Big)^{1/2}\Big(1-\frac{q^2}{r^2}\Big) - \frac{r}{2}\sqrt{r^2+q^2} - \frac{q^2}{2}\ln|r+\sqrt{r^2+q^2}| + 2q^2\ln|r+\sqrt{r^2+q^2}|\Big]_{r_o}^{r_+}
\end{aligned}$$

On rearrangement,

$$\boxed{S = \pi r^2\Big[\Big(1+\frac{q^2}{r^2}\Big)^{1/2}\Big(1-\frac{2q^2}{r^2}\Big) + \frac{3q^2}{r^2}\ln|r+\sqrt{r^2+q^2}|\Big]_{r_o}^{r_+}} \tag{7.3}$$

7.3 Volume

$$V = \frac{\partial M}{\partial P}\Big|_{S,q}$$

Differentiating (7.1) with respect to AdS radius l,

$$\begin{aligned}
\frac{\partial M}{\partial l} &= (r^2+q^2)^{3/2}\Big[\frac{2l}{2l^2r^2} - \frac{2(l^2+r^2)}{2l^3r^2}\Big] \\
&= -(r^2+q^2)^{3/2}\Big(\frac{1}{l^3}\Big)
\end{aligned} \tag{7.4}$$

The pressure of the AdS black in 4-dimension is given by[KM12]

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}$$

Differentiating the above equation with respect to AdS radius l,

$$\frac{\partial P}{\partial l} = -\frac{3}{4\pi l^3} \tag{7.5}$$

Using (7.4) and (7.5),

$$V = \frac{\partial M/\partial l}{\partial P/\partial l}$$

$$\boxed{V = \frac{4}{3}\pi r^3\Big(1+\frac{q^2}{r^2}\Big)^{\frac{3}{2}}} \tag{7.6}$$

7.4 Potential

$$\phi = \frac{\partial M}{\partial q}\Big|_{S,P}$$

Differentiating (7.1) with respect to magnetic monopole charge q,

$$\boxed{\phi = \frac{3q(l^2+r^2)(q^2+r^2)^{\frac{1}{2}}}{2l^2r^2}} \tag{7.7}$$

7.5 Using Smarr Relation

Consider mass M as a function of entropy S, pressure P and magnetic monopole charge q. Then

$$\begin{aligned} M &= M(S,P,q) \\ dM &= \frac{\partial M}{\partial S}\Big|_{P,q} dS + \frac{\partial M}{\partial P}\Big|_{S,q} dP + \frac{\partial M}{\partial q}\Big|_{S,P} dq \end{aligned}$$

where

$$\frac{\partial M}{\partial S}\Big|_{P,q} = T \qquad \frac{\partial M}{\partial P}\Big|_{S,q} = V \qquad \frac{\partial M}{\partial q}\Big|_{S,P} = \phi$$

Thus the first law of thermodynamics is rewritten as

$$dM = TdS + VdP + \phi dq$$

Using Euler's Homogeneous function theorem, the above equation can be rewritten as

$$M = 2TS' - 2PV + \phi\ q$$

This is the Smarr relation for the stationary black holes. Now the entropy is written as

$$S' = \frac{M + 2PV - \phi\ q}{2T}$$

$$\begin{aligned} S' &= \frac{\frac{(l^2+r^2)(r^2+q^2)^{3/2}}{2l^2r^2} + 2\left(\frac{3}{8\pi l^2}\right)\frac{4\pi r^3}{3}\left(1+\frac{q^2}{r^2}\right)^{3/2} - \frac{3q^2(l^2+r^2)(q^2+r^2)^{1/2}}{2l^2r^2}}{2\frac{3r^4+l^2(r^2-2q^2)}{4\pi l^2 r(q^2+r^2)}} \\ &= \frac{(q^2+r^2)^{3/2}}{2l^2r^2}\frac{\frac{3r^4+l^2(r^2-2q^2)}{q^2+r^2}}{\frac{3r^4+l^2(r^2-2q^2)}{2\pi l^2 r(q^2+r^2)}} \end{aligned}$$

On rearrangement, the entropy S′ using smarr relation becomes

$$S' = \pi r^2\left(1+\frac{q^2}{r^2}\right)^{3/2} \tag{7.8}$$

Then the extra pressure could be written as

$$P_q = -\frac{1}{8\pi q^2}$$

and the coupled volume becomes

$$V_q = \frac{\partial M}{\partial P_q}\Big|_{S,P} = \frac{\partial M/\partial q}{\partial P_q/\partial q}$$

$$\frac{\partial M}{\partial q} = \frac{3q(l^2+r^2)\sqrt{q^2+r^2}}{2l^2r^2}$$

$$\frac{\partial P_q}{\partial q} = \frac{1}{4\pi q^3}$$

$$V_q = \frac{\partial M/\partial q}{\partial P_q/\partial q} = \frac{3q^2(l^2+r^2)\sqrt{1+\frac{q^2}{r^2}}}{2l^2r^2}4\pi q^3$$

$$V_q = \frac{6\pi q^5}{r^2}\sqrt{1+\frac{q^2}{r^2}}\Big(1+\frac{r^2}{l^2}\Big) \tag{7.9}$$

7.6 Equation of State

Considering the equation of temperature,

$$\begin{aligned} T &= \frac{3r^4+l^2(r^2-2q^2)}{4\pi l^2 r(q^2+r^2)} \\ &= \frac{3r^4}{4\pi l^2 r(q^2+r^2)} + \frac{l^2(r^2-2q^2)}{4\pi l^2 r(q^2+r^2)} \\ &= \Big(\frac{3}{8\pi l^2}\Big)\frac{2r^3}{(r^2+q^2)} + \frac{(r^2-2q^2)}{4\pi r(q^2+r^2)} \\ &= P\frac{2r^3}{(r^2+q^2)} + \frac{r^2}{4\pi r(q^2+r^2)} - 2\frac{q^2}{4\pi r(q^2+r^2)} \end{aligned}$$

On rearrangement,

$$\begin{aligned} P\frac{2r^3}{(r^2+q^2)} &= T - \frac{r}{4\pi(q^2+r^2)} + \frac{q^2}{2\pi r(q^2+r^2)} \\ 2r^3P &= \frac{4\pi rT - r^2 + 2q^2}{4\pi r} \\ P &= \frac{4\pi rT - r^2+2q^2}{8\pi r^4} \end{aligned}$$

From (7.6) we have

$$\begin{aligned} V &= \frac{4}{3}\pi r^3\Big(1+\frac{q^2}{r^2}\Big)^{\frac{3}{2}} \\ (r^2+q^2)^{3/2} &= \frac{3V}{4\pi} \\ r^2+q^2 &= \Big(\frac{6V}{8\pi}\Big)^{2/3} \\ r^2 &= \frac{1}{4}\Big(\frac{6V}{\pi}\Big)^{2/3} - q^2 \\ r &= \frac{1}{2}\sqrt{\Big(\frac{6V}{\pi}\Big)^{2/3} - 4q^2} \end{aligned}$$

Substituting the value of r in the above equation,

$$P = \frac{4\pi\frac{1}{2}\sqrt{\left(\frac{6V}{\pi}\right)^{2/3} - 4q^2}\ T - \frac{1}{4}\left(\frac{6V}{\pi}\right)^{2/3} + q^2 + 2q^2}{8\pi\left[\frac{1}{4}\left(\frac{6V}{\pi}\right)^{2/3} - q^2\right]^2}$$

On rearrangement,

$$\boxed{P = \frac{12q^2 + \left(\frac{6V}{\pi}\right)^{2/3}\left[-1 + 2\pi T\sqrt{\left(\frac{6V}{\pi}\right)^{2/3} - 4q^2}\right]}{2\pi\left[\left(\frac{6V}{\pi}\right)^{2/3} - 4q^2\right]^2}} \tag{7.10}$$

The equation (7.10) is called **Equation of State**.

7.7 P-V Criticality

The point where phase equilibrium curve terminates is called critical point and there will be no difference of phases beyond this point. Thus at critical point, two phase become identical. Thus the isotherm has a point of inflection at this point.

This satisfies the conditions,

$$\frac{\partial P}{\partial V} = 0 = \frac{\partial^2 P}{\partial^2 V}$$

Differentiating (7.10) with respect to the volume,

$$\begin{aligned}\frac{\partial P}{\partial V} &= \frac{(0.0689278v^{2/3} - 0.895551q^2)}{(v^{1/3}(v^{2/3} - 2.59852q^2)^3)} \\ &\quad + T\frac{(-0.866173q^2v^{2/3}) + ((6/\pi)^{2/3}v^{2/3} - 4q^2)(-1.12538q^2 - 0.433086v^{2/3}) + 0.333333v^{4/3})}{v^{1/3}(v^{2/3} - 2.59852q^2)^3((6/\pi)^{2/3}v^{2/3} - 4q^2)^{1/2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 P}{\partial^2 V} &= -\frac{0.11488 + \frac{0.775704q^4}{v^{4/3}} - \frac{2.02992q^2}{v^{2/3}} - \frac{2.51304*10^-16q^4)}{(-4q^2+(6/\pi)^{2/3}v^{2/3})v^{2/3})}}{(45.5935q^8 - 70.1839q^6v^(2/3) + 40.5138q^4v^(4/3) - 10.3941q^2v^2 + v^(8/3)))} \\ &\quad - T\Bigg[- 0.721811(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{0.5} + q^2\Bigg(- \frac{0.577448}{(-4q^2 + (6/\pi)^{2/3}v^{2/3}))^{0.5}} \\ &\quad - \frac{(3.00102(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{0.5}}{v^{2/3}} - \frac{0.888889v^{2/3}}{(-4q^2 + (6/\pi)^{2/3}v^{2/3}))^{1.5}}\Bigg) + \frac{0.555556v^{2/3}}{(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{0.5}} \\ &\quad + \frac{0.171038v^{4/3}}{(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{1.5}} + \frac{1}{v^{4/3}}1.1549q^4\Bigg(0.844037(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{0.5} \\ &\quad - \frac{1.94889v^{2/3}}{(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{0.5}} + \frac{v^{4/3}}{(-4q^2 + (6/\pi)^{2/3}v^{2/3})^{1.5})}\Bigg)\Bigg] \\ &\quad \Big/ (45.5935q^8 - 70.1839q^6v^{2/3} + 40.5138q^4v^{4/3} - 10.3941q^2v^2 + v^{8/3})\end{aligned}$$

Dividing the above 2 equations gives,

$$V_c \simeq 287.44\ q^3 \tag{7.11}$$

Substituting (7.11) in one of the above equation gives,

$$T_c \simeq \frac{0.0251}{q} \tag{7.12}$$

and thus gives

$$P_c \simeq \frac{0.0012}{q^2} \tag{7.13}$$

The dimensionless quantities are defines as

$$p = \frac{P}{P_c} \qquad v = \frac{V}{V_c} \qquad \mathcal{T} = \frac{T}{T_c}$$

Substituting it in the equation of state,

$$p\frac{0.0012}{q^2} = \frac{12q^2 + \left(\frac{1724.64\ q^3}{\pi}\right)^{2/3}\left[-1 + 2\pi\frac{0.0251}{q}\sqrt{\left(\frac{1724.64\ q^3}{\pi}\right)^{2/3} - 4q^2}\right]}{2\pi\left[\left(\frac{1724.64\ q^3}{\pi}\right)^{2/3} - 4q^2\right]^2}$$

$$\boxed{p = \frac{0.354 + v^{2/3}\left(-1.978 + 0.312\mathcal{T}\sqrt{-4 + 67.045v^{2/3}}\right)}{(0.06 - v^{2/3})^2}} \tag{7.14}$$

We denote

$$\omega = \frac{V - V_c}{V_c} = v - 1 \qquad t = \frac{T - T_c}{T_c} = \mathcal{T} - 1$$

The above equation is rewritten as

$$p = \frac{0.354 + (1+\omega)^{2/3}\left(-1.978 + 0.312(1+t)\sqrt{-4 + 67.045(1+\omega)^{2/3}}\right)}{(0.06 - (1+\omega)^{2/3})^2}$$

Expanding and approximating (3.14),

$$\boxed{p \approx 0.965 + 2.802t - 1.112\omega t - 0.054\omega^3} \tag{7.15}$$

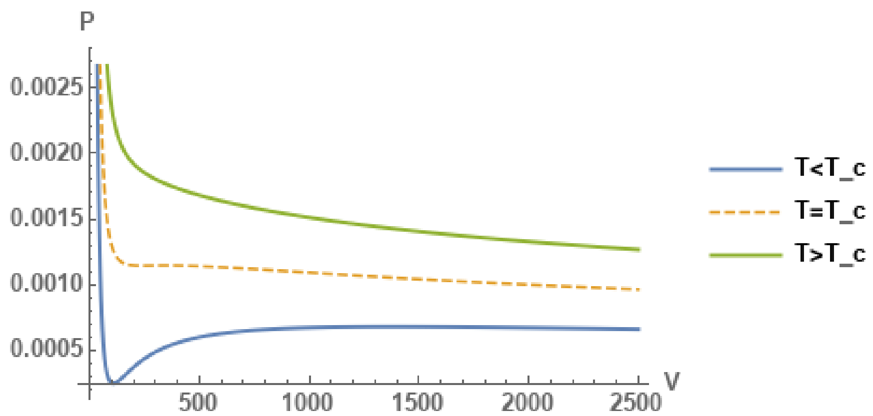

Figure 7.3: The isotherms of BAdS black hole by considering q=1 where T_c=0.025

In the above graph, when T > T_c the isotherm decreases monotonically. When T = T_c the isotherm has a inflection point. When T > T_c there exist minimum and maximum. This graph is identical to the real gas P-V diagram.

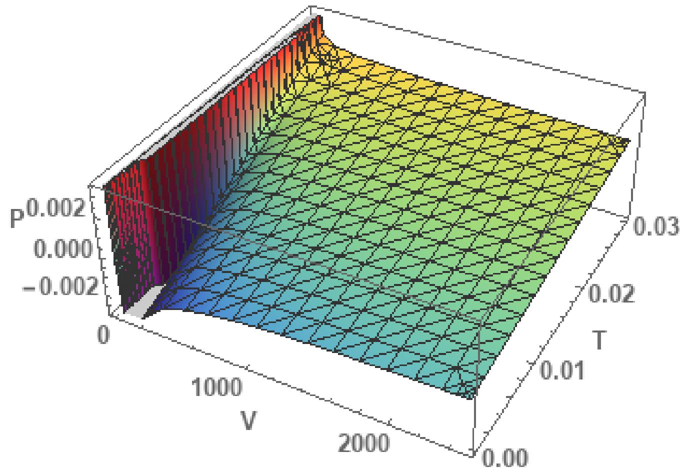

Figure 7.4: The 3D map of P-V-T Diagram

7.8 Specific Heat Capacity

7.8.1 At constant Volume

Specific heat capacity at constant volume is defined by

$$C_v = T\left(\frac{\partial S}{\partial T}\right)_{V,q} = T\left(\frac{\partial S/\partial r}{\partial T/\partial r}\right)_{V,q}$$

Since we know,

$$r = \frac{1}{2}\sqrt{\left(\frac{6V}{\pi}\right)^{2/3} - 4q^2}$$

At constant V and q, r also remains as a constant and that gives

$$\frac{\partial S}{\partial r} = 0$$

Thus $C_V = 0$

7.8.2 At constant Pressure

Specific heat capacity at constant pressure is defined by

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_{P,q} = T\left(\frac{\partial S/\partial r}{\partial T/\partial r}\right)_{P,q}$$

$$\frac{\partial S}{\partial r_+} = 2\pi\frac{(q^2+r_+^2)^{3/2}}{r_+^2}$$
$$\frac{\partial T}{\partial r_+} = \frac{2r_+ + 12r_+^3}{4\pi r_+(q^2+r_+^2)} - \frac{-2q^2+r_+^2+3r_+^4}{2\pi(q^2+r_+^2)^2} - \frac{-2q^2+r_+^2+3r_+^4}{4\pi r_+^2(q^2+r_+^2)}$$

$$C_p = 2\pi \frac{3r^4 + l^2(r^2 - 2q^2)}{4\pi l^2 r(q^2 + r^2)} \frac{(q^2 + r_+^2)^{3/2}}{r_+^2} \Bigg/$$
$$\frac{2r_+ + 12r_+^3}{4\pi r_+(q^2 + r_+^2)} - \frac{-2q^2 + r_+^2 + 3r_+^4}{2\pi(q^2 + r_+^2)^2} - \frac{-2q^2 + r_+^2 + 3r_+^4}{4\pi r_+^2(q^2 + r_+^2)}$$
$$C_p = \frac{2\pi(3r_+^4 + l^2(r_+^2 - 2q^2))(q^2 + r_+^2)^{5/2}}{3r_+^5(3q^2 + r_+^2) + l^2 r_+(2q^4 + 7q^2 r_+^2 - r_+^4)}$$

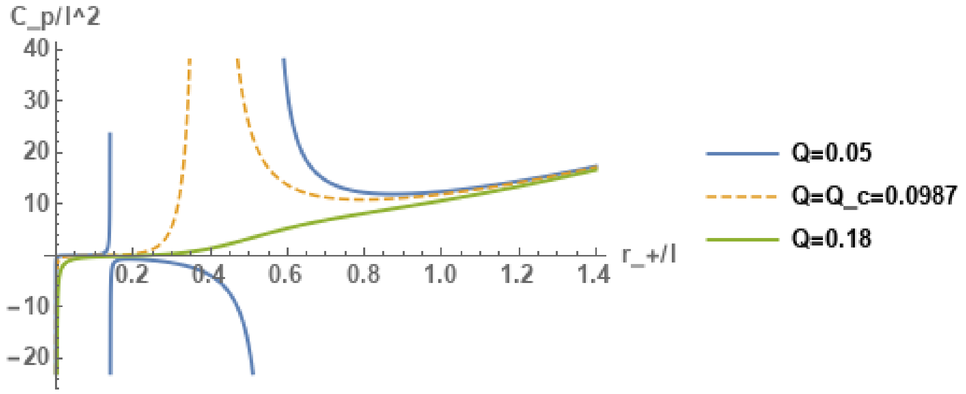

Figure 7.5: The graph of Specific heat capacity at constant pressure

In the above, there are 3 cases.

- When Q < Q_c(Q = 0.05), there are three branches.
  1. When r_+/l is approximately between 0.0702 and 0.14, $C_p > 0$. Thus BAdS black hole is locally stable within this region.
  2. When r_+/l is approximately between 0.14 and 0.55, $C_p < 0$. This intermediate BAdS black hole is unstable stable within this region.
  3. For r_+/l more than 0.55, $C_p > 0$.

  Let $r_s/l \simeq 0.14$ be the radius of the event horizon of small BAdS black hole and $r_l/l \simeq 0.55$ be the radius of the event horizon of large BAdS black hole.
  From the above description, it is clear that the phase transition occurs at r_s/l and r_l/l where the C_p value diverges. This shows there exist first order phase transition between small and large stable black hole.

- When Q ≈ Q_c(Q =0.0987), there are two branches. These two branches merge and have a maximum value at $r_+/l = 0.392$. In turn, the small and large black hole coexist at r_+/l = 0.392 where $C_p \to \infty$. This point is called inflection point.

- When Q > Q_c(Q = 0.18), there exist only one curve which shows C_p is positive always. Thus there is only one stable black hole. In turn, there is no phase transition.

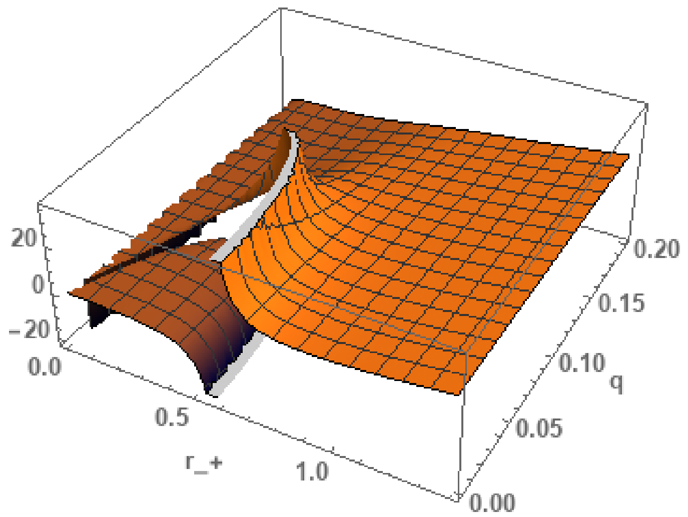

Figure 7.6: The 3D map of Specific heat capacity at constant pressure

7.9 Gibbs free energy

The Gibbs free energy is defined as

$$\begin{aligned} G &= M - TS \\ &= \frac{(l^2 + r_+^2)(r_+^2 + q^2)^{3/2}}{2l^2 r_+^2} - \pi r_+^2 \frac{3r_+^4 + l^2(r_+^2 - 2q^2)}{4\pi l^2 r(q^2 + r_+^2)} \\ &= \frac{(l^2 + r_+^2)(r_+^2 + q^2)^{3/2}}{2l^2 r_+^2} - r_+ \frac{3r_+^4 + l^2(r_+^2 - 2q^2)}{4l^2(q^2 + r_+^2)} \end{aligned}$$

where M is the mass of the BAdS black hole, T is the Hawking Temperature, S is the entropy of the black hole, q is the charge of the black hole, and r_+ is the radius of the event horizon.

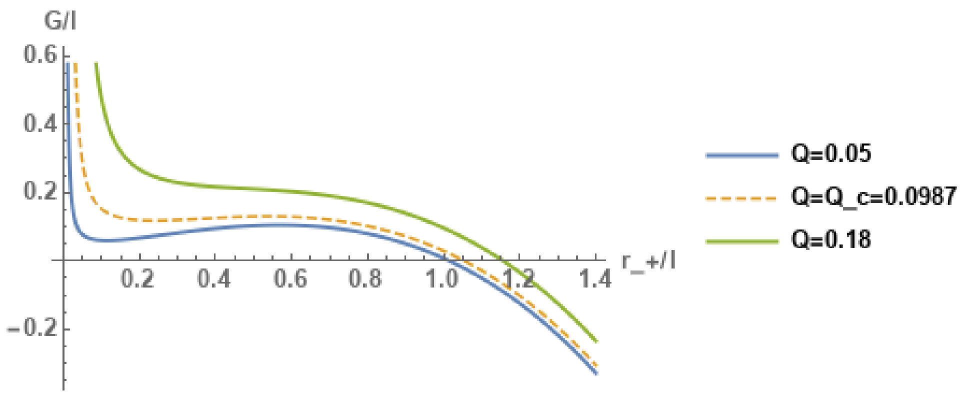

Figure 7.7: Graph of Gibbs free energy with the variation of BAdS black hole event horizon radius

In the above graph, all three curves cross the horizontal axis i.e., G value changes from positive to negative. The Gibbs free energy becomes very large when the radius of event horizon is very small and vice versa i.e., Gibbs free energy decreases with increase of r_+. The system is stable when Gibbs free energy is small. Gibbs free energy is small when entropy is a positive quantity. In turn, it must decrease with temperature. Thus large BAdS black hole is stable.

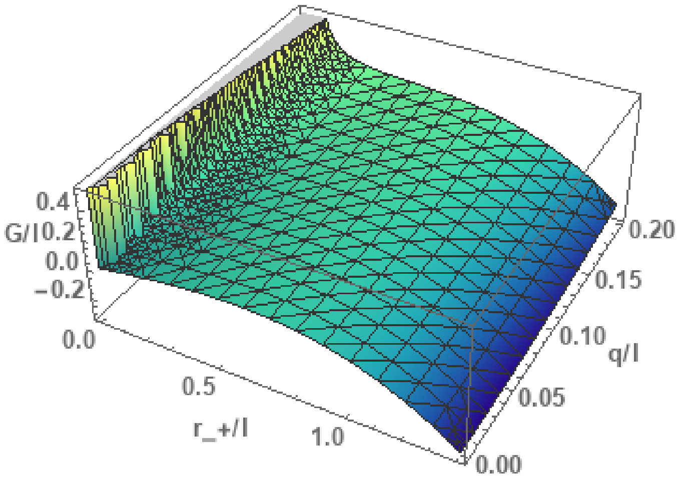

Figure 7.8: The 3D map of Gibbs free energy with respect to charge and radius of BAdS black hole

7.10 The Critical Exponents

Many thermodynamic system have a anomalous behaviour near the critical point. The theory of phase transition is to study this anomalous behaviour in the vicinity of critical point. At the critical point, the physical quantity X can be written using power laws.

$$X \propto (Y - Y_c)^m$$

Where m is called the critical exponent.

7.10.1 α

C_v shows the weak divergence at the critical point. Thus the singular behaviour of the specific heat at constant volume C_v can be return as

$$C_v = T\frac{\partial S}{\partial T}\Big|_v \propto |t|^{-\alpha}$$

Since $C_v = 0$, it is independent of —t—. Thus, α=0.

7.10.2 β

The difference between the volume of the large and small black hole vanishes at the critical point.

$$\eta = V_l - V_s \propto |t|^\beta \tag{7.16}$$

Differentiating (7.15) with respect to ω

$$\frac{dp}{d\omega} = -1.112t - 0.162\omega^2$$

Using Maxwell's area law we have,

$$\int \omega dp = 0$$

$$\int_{\omega_s}^{\omega_l} \omega(-1.112t - 0.162\omega^2)d\omega = 0$$

$$\int_{\omega_s}^{\omega_l} (1.112\omega t + 0.162\omega^3)d\omega = 0$$

$$0.556\omega_l^2 t + 0.0405\omega_l^4 - \big(0.556\omega_s^2 t + 0.0405\omega_s^4\big) = 0$$

$$0.0405(\omega_l^4 - \omega_s^4) + 0.556t\big(\omega_l^2 - \omega_s^2\big) = 0$$

This gives ω_l = -ω_s.
Equation (7.15) can also be written as

$$\begin{aligned} p &= 0.965 + 2.802t - 1.112\omega_l t - 0.054\omega_l^3 \\ &= 0.965 + 2.802t - 1.112\omega_s t - 0.054\omega_s^3 \end{aligned}$$

$$0.965 + 2.802t - 1.112\omega_l t - 0.054\omega_l^3 - (0.965 + 2.802t - 1.112\omega_s t - 0.054\omega_s^3) = 0$$

The solution of the above equation is,

$$\begin{aligned} \omega_l &= 0.0555556(-9\omega_s - 1.73205\sqrt{-2224t - 81\omega_s^2}) \\ &= -0.5\omega_s - 0.09622\sqrt{-2224t - 81\omega_s^2} \end{aligned}$$

Substituting ω_l = -ω_s gives,

$$\begin{aligned} \omega_l = -\omega_s &\simeq 4.43791i\sqrt{t} \\ &\simeq 4.43791\sqrt{-t} \end{aligned}$$

Thus (7.16) can be written as

$$\begin{aligned} \eta &= V_l - V_s \\ &= V_c(\omega_l - \omega_s) \\ &= 8.87582V_c\sqrt{-t} \end{aligned}$$

Thus by comparison, β = 1/2.

7.10.3 γ

The isothermal compressibility is given by the equation,

$$\kappa_T = -\frac{1}{T}\frac{\partial V}{\partial P}\bigg|_T \propto |t|^{-\gamma} \tag{7.17}$$

At the critical point, $\frac{\partial V}{\partial P}$ becomes infinity, since $\frac{\partial P}{\partial V}$ is zero. Thus the isothermal compressibility becomes infinity.
Using above tranformation, κ_T can be rewritten as

$$\begin{aligned} \kappa_T &= -\frac{1}{(1+\omega)}\frac{1}{P_c}\frac{d\omega}{dp} \\ &= \frac{1}{P_c}\frac{1}{(1+\omega)(1.112t + 0.162\omega^2)} \\ &\propto \frac{1}{t} \end{aligned}$$

Thus by comparison, γ = 1.

7.10.4 δ

In the P-V diagram, at T = T_c the critical isotherm has a horizontal tangent at the critical point.

$$|P - P_c| = |V - V_c|^\delta \tag{7.18}$$

This can be rewritten as

$$p - 1 = \omega^\delta$$

But at T = T_c,

$$t = \frac{T - T_c}{T_c} = 0$$

Thus setting t=0, (7.15) can be written as

$$p = 0.965 - 0.054\omega^3$$

$$p = -0.035 - 0.054\omega^3$$

Thus the value of δ= 3.
The critical exponents α, β, γ and δ of BAdS black hole is 0,1/2,1,3 respectively.This is identical to Vander Waals gas.

Chapter 8

Conclusion

The anti-de Sitter space is a maximally symmetric space having negative constant curvature.The metric we got is similar to that of Lobechevski space.We have proved that the entire AdS space can also be called as universal covering space.Then the causal properties of AdS space is studied using Conformal Diagram.Even though the conformal diagram of AdS space is compact, the causal properties are well studied using Poincare coordinates with the time coordinate range $-\infty < t < \infty$. Then we have studied Reissner-Nordst$\ddot{o}$rm black hole and the conformal diagram is drawn.Using the knowledge of RN black hole and AdS space,equation of state of RN-AdS black hole is obtained. Pressure,temperature and volume corresponding to the inflection point are calculated.Thus P-V graph is plotted. Using the obtained result it is concluded that RN-AdS black hole behaves like a Vander Waals fluid.

Later considering Bardeen black hole, equations of motion is calculated using the action intergral. With the help of action interal the detailed study of Hawking temperature is done. T-r_+ graph is plotted. Using the equation of state, P-V diagram is plotted. Critical pressure, critical volume, critical temperature is calculated. Transition from smaller black hole to larger black hole is examined using the detailed study of specific heat capacity and Gibbs free energy. It is observed that the BAdS black hole shows first order phase transition. At last, critical exponents are calculated. The values of α, β, γ and δ is equivalent to the critical exponent values of Vander Waals fluid.

Here conformal diagrams are drawn using *mathcha* and the graphs are drawn using *mathematica online.*

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