The Spinorial Formulations of Scattering Amplitudes and Loop Quantum Gravity

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by

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Certificate

This is to certify that this dissertation entitled **The Spinorial Formulations of Scattering Amplitudes and Loop Quantum Gravity** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Devadharsini.S at Indian Institute of Science Education and Research under the supervision of Dr.Deepak Vaid, Professor, Department of Physics, NIT-K, during the academic year 2020-2021.

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This thesis is dedicated to my mother, whose love and support have been unwavering through all the tough times.

Declaration

I hereby declare that the matter embodied in the report entitled **The Spinorial Formulations of Scattering Amplitudes and Loop Quantum Gravity** are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr.Deepak Vaid and the same has not been submitted elsewhere for any other degree.

Devadharsini.S

Date : July 30, 2021

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Abstract

We study the Spinor Helicity formalism used to represent scattering data in the calculation of scattering amplitudes and the spinorial formulation of Loop Quantum Gravity and establish a correspondence between the two. We show that the kinematic space of the scattering of N massless particles is represented by the complex Grassmannian G(2,N) which also represents the set of U(N) invariant coherent states defined by Friedel and Livine. We further show that our correspondence implies the correspondence between the closure constraint, the Gauss's law in the intertwiner space and the conservation of the spatial components of momenta in the scattering amplitude picture. We further show that this correspondence implies that the total energy of the scattering amplitude corresponds to the total area of the coherent state.

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1 Motivation

In QFT, scattering amplitudes are calculated as a perturbation series in the expansion of a small dimensionless parameter called the coupling constant. In the late 1940s, Feynman introduced a diagrammatic way to organize the perturbative calculation of the scattering amplitudes. It expresses the n-particle scattering amplitude A_n as a sum of all possible Feynman diagrams with n external legs. We can convert a Feynman diagram to a mathematical expression by using the Feynman Rules.

But one of the major drawbacks of using this method is that the number of Feynman diagrams that need to be summed over tends to grow very quickly with the number of particles involved.

For example, consider tree level gluon scattering amplitudes in QCD,

$$\begin{aligned} g+g &\longrightarrow g+g && \text{4 diagrams} \\ g+g &\longrightarrow g+g+g && \text{25 diagrams} \\ g+g &\longrightarrow g+g+g+g && \text{220 diagrams} \end{aligned}$$

and the $g+g \longrightarrow 8g$ has more than one million diagrams [9]. Even the mathematical expression for each diagram becomes much more complicated as the number of external particles grows.

The premise of the papers [2] [6] on the Amplituhedron by Nima Arkani-Hamed is that the scattering amplitudes of the $\mathcal{N}=4$ Super Yang Mills theory can be calculated without resorting to the painstaking calculations involving Feynman diagrams. We define a geometric object called a positive geometry and its associated canonical form. The Amplituhedron is a positive geometry whose associated canonical form can be used to read off scattering amplitudes of the theory.

The Amplituhedron formalism in Nima's work is constructed using Momentum Twistors, which are constructed from the Spinor Helicity variables. What we aim to do is to apply these concepts from the Amplituhedron formalism to Loop Quantum Gravity (LQG), in an attempt to calculate scattering amplitudes in LQG. This, we believe, would help us further establish LQG as a valid contender for a theory of quantum gravity.

As a first step in this process, we have established a correspondence between the Spinor Helicity formalism of calculating scattering amplitudes and the spinorial formulation of LQG and shown that the Friedel-Levine coherent states [11] of N-valent intertwiners can be thought of as states in the kinematic space of an N-particle scattering amplitude.

Part I

Scattering Amplitudes

2 Lorentz Algebra

The Lorentz group is the group of all transformations that keep the line element given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1}$$

invariant. It is the group SO(1,3). Its elements $\Lambda \in SO(1,3)$ satisfy,

$$\Lambda\eta\Lambda = \eta \tag{2.2}$$

This group has six generators with the following commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \; ; \quad [J_i, K_j] = i\varepsilon_{ijk}K_k \; ; \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k \tag{2.3}$$

The rotation generators the J_i are hermitian, and the boost generators K_i are anti-hermitian. We define,

$$J_i^{\pm} = \frac{1}{2}(J_i \pm K_i) \tag{2.4}$$

where the $J_i^{\pm}$'s are now hermitian. The Lorentz algebra now becomes

$$[J_i^{\pm}, J_j^{\pm}] = i\varepsilon_{ijk}J_k^{\pm} \; ; \quad [J_i^{\pm}, J_j^{\mp}] = 0 \tag{2.5}$$

This shows that the Lorentz algebra is equivalent to two SU(2) algebras. Thus the Lorentz algebra is a specific real form of SO(4). For SO(4), the third commutation relation in (2.3) has a plus sign in the RHS. In Euclidean space, all representations are real or pseudoreal, but in Minkowski space complex conjugation interchanges the two SU(2)'s. This can also be seen at the level of the generators $J_i^{\pm}$. In order for all rotation and boost parameters to be real, one must take all the J_i's and K_i's to be imaginary and hence from (2.4) we can see that

$$(J_i^{\pm})^* = -J_i^{\mp} \tag{2.6}$$

In summary, we have [7]

$$SO(4) \cong SU(2) \times SU(2) \tag{2.7}$$

$$SO(1,3) \cong SU(2) \times SU(2)^* \tag{2.8}$$

where $SU(2)^* = \{A^* \mid A \in SU(2)\}$.

3 Introduction to Scattering Amplitudes

Wigner defines a particle as an irreducible unitary representation of the Lorentz group.

Consider a 4-momentum p^μ. The action of the (proper orthochronus) Lorentz group on p^μ is given by, $\Lambda^\mu_\nu k^\nu$. This action yields 6 orbits. In our signature $(-1,+1,+1,+1)$, these 6 orbits are given by,

$$
\begin{aligned}
&(i)\ p^\mu = 0 = (0,0,0,0) \longrightarrow \text{This orbit has only one element.}\\
&(ii)\ p^2 > 0 \longrightarrow \text{Tachyon (unphysical state)}\\
&\left.\begin{aligned}&(iii)\ p^2 = 0 \text{ such that } p^0 > 0\\ &(iv)\ p^2 = 0 \text{ such that } p^0 < 0\end{aligned}\right\} \text{massless particles}\\
&\left.\begin{aligned}&(v)\ p^2 < 0 \text{ such that } p^0 > 0\\ &(vi)\ p^2 < 0 \text{ such that } p^0 < 0\end{aligned}\right\} \text{massive particles}
\end{aligned}
$$

We work with cases (iii) and (v).

Any 4-momentum p^μ can be brought into a standard form given by,

$$
k^\mu = \begin{cases} (k^0,0,0,k^0) & \text{for massless particles} \\ (k^0,0,0,0) & \text{for massive particles} \end{cases}
$$

For a massive particle this can be done by going to the rest frame, and for a massless particle, this can be done by rotating the frame of reference to get the z-axis aligned with the direction of momentum.

The Little group is defined as the set of Lorentz transformations that preserve k^μ. Thus, for a massive particle, the little group is $SO(3)$ and for a massless particle, the little group is $U(1)$.

(In fact, Wigner defines a particle as an irreducible unitary representation of the Poincaré group. We consider them to only be representations of the Lorentz group, because the Lorentz group is what we focus on. The Little group is also defined as the set of Poincaré transformations that preserve k^μ. Thus, for a massive particle, the little group is $SO(3)$, it does not change. But, a massless particle, apart from rotations about the z-axis, can also be translated in the plane perpendicular to the z-axis. Thus, for a massless particle, the little group is in fact $U(1) \times \mathbb{R}^2$ and not just $U(1)$ like we have taken.)

4 Spinor Helicity Formalism

4.1 Introduction

Now, generally, the external data of a scattering amplitude is given in terms of the momenta of the n scattering particles, labelled p_i, where $i = 1,2,...,n$. As a convention we consider all of them to be outgoing particles.

Thus, from momentum conservation, we get

$$\sum_i p_i^\mu = 0 \tag{4.1}$$

In this report, we work only with scattering amplitudes of massless bosons. In this case, all the momenta are null,

$$p_i^2 = 0 \tag{4.2}$$

Equations (4.1) and (4.2) are the two constraints on the external data of the scattering of n massless particles.

There are multiple ways to represent the external data of a scattering amplitudes :

1) Spinor Helicity variables

2) Mandelstam Invariants

3) Momentum Twistors

The Spinor Helicity variables make the constraint (4.2) manifest. The Mandelstam Invariants make the constraint (4.1) manifest. The Momentum Twistors make the constraints (4.1) and (4.2) manifest. We will be using the Spinor Helicity variables in this report.

4.2 Formalism

We can write an external momentum p as a 2×2 matrix, as follows (where a and $\dot{b}$ are the matrix indices, that is, $a, \dot{b} = 1, 2$),

$$\begin{aligned} p_{a\dot{b}} &= (p_\mu \sigma^\mu)_{a\dot{b}} \qquad\qquad \text{where } \sigma^\mu = (1, \sigma^i) \\ &= \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \end{aligned} \tag{4.3}$$

$$det[p_{a\dot{b}})] = p^2 = 0 \tag{4.4}$$

A 2×2 matrix with $det = 0$ has rank atmost 1. This implies that it is possible to express p as an outer product of two 2-component objects [8], which we will call $|p]_a$ and $\langle p|_{\dot{b}}$.

$$p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}} \tag{4.5}$$

The two 2-component objects, $|p]_a$ and $\langle p|_{\dot{b}}$ are called Spinor Helicity variables. They make the constraint in (4.2) manifest.

The symmetry group of p^μ is the Lorentz group SO(1,3). The two spinors $|p]_a$ and $\langle p|_{\dot{b}}$ that make up the matrix $(p)_{a\dot{b}}$ belong to the representations of $SU(2)$ and $SU(2)^*$ respectively. The undotted indices a and b are $SU(2)$ indices and the dotted indices are $SU(2)^*$ indices.

We also have,

$$p^{\dot{a}b} = (p_\mu \bar{\sigma}^\mu)^{\dot{a}b} \qquad\qquad \text{where } \bar{\sigma}^\mu = (1, -\sigma^i) \tag{4.6}$$

represented by the spinor helicity variables,

$$p^{\dot{a}b} = -|p\rangle^{\dot{a}} [p|^b \tag{4.7}$$

Since these spinors belong to the $SU(2)$ and $SU(2)^*$ algebras, their indices are raised and lowered using the epsilon tensor.

$$[p|^a = \varepsilon^{ab} |p]_b \quad ; \quad |p\rangle^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}} \langle p|_{\dot{b}} \tag{4.8}$$

where

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.9}$$

Now, recall the Weyl equations given by,

$$\not{p}\, v_{\pm}(p) = 0 \quad \text{and} \quad \bar{u}_{\pm}(p)\, \not{p} = 0 \tag{4.10}$$

Recall that we also have the relations $v_{\pm} = u_{\mp}$ and $\bar{u}_{\pm} = \bar{v}_{\mp}$. Our spinor helicity variables can be used to represent the solutions of the Weyl equation as [9],

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \tag{4.11}$$

$$\bar{u}_- = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix} \qquad \bar{u}_+ = \begin{pmatrix} [p|^a & 0 \end{pmatrix} \tag{4.12}$$

The spinor completeness relation gives us,

$$-\not{p} = |p\rangle\,[p| + |p]\,\langle p| \tag{4.13}$$

Thus, these our spinor helicity variables are spinors that belong to the right handed and the left handed SU(2) algebras, (the $SU(2)$ and $SU(2)^*$ algebras are the $SU(2)_L$ and $SU(2)_R$ algebras respectively). Thus, the square spinors are left handed Weyl spinors and the angle spinors are right handed Weyl spinors.

Thus, every null momenta p_i is denoted by two spinors $|p_i\rangle$ and $|p_i]$. In order to simplify the notation, we denote $|p_i\rangle$ and $|p_i]$ as $|i\rangle$ and $|i]$ respectively.

Now, for two null vectors p^μ and k^μ, we define a Lorentz invariant inner product for $SU(2)_L$ and $SU(2)_R$ as follows,

$$\langle pk\rangle = \langle p|_{\dot{a}}\, |k\rangle^{\dot{a}} \quad ; \quad [pk] = [p|^a |k]_a \tag{4.14}$$

These brackets are antisymmetric due to the antisymmetric nature of the epsilon tensor.

$$\langle pk\rangle = -\langle kp\rangle \quad ; \quad [pk] = -[kp] \tag{4.15}$$

The momentum conservation condition $\sum_i p_i^\mu = 0$ in the spinor helicity variables is given by,

$$\sum_i |i\rangle\,[i| = 0 \tag{4.16}$$

Thus, for two null vectors k and q, equation (4.16) implies that

$$\sum_i \langle qi\rangle [ik] = 0 \tag{4.17}$$

We can also derive,

$$\langle pk\rangle [pk] = -2p\cdot k = (p+k)^2 \tag{4.18}$$

Now, we define the angle-square bracket as $\langle p|\,\gamma^\mu |k]$ and the square-angle bracket $[p|\gamma^\mu\,|k\rangle$ as the contraction of the two 2-component spinors with the $4\times 4\gamma$ matrix. These brackets have the property,

$$[k|\gamma^\mu\,|p\rangle = \langle p|\,\gamma^\mu |k] \tag{4.19}$$

We have the Fierz identity, given by,

$$\langle p_1|\gamma^\mu|p_2]\langle p_3|\gamma_\mu|p_4] = 2\langle p_1 p_3\rangle[p_2 p_4] \tag{4.20}$$

Now, consider a vector V_μ. Then we define,

$$\langle p|V|k] = V_\mu \langle p|\gamma^\mu|k] \tag{4.21}$$

Now, if V_μ is null, then we have the identity,

$$\langle p|V|k] = -\langle pV\rangle[Vk] \tag{4.22}$$

4.3 Real Momenta

To keep things general, we work with complex momentum, and impose the condition that momentum is real in the last step. This is because, working with complex momenta gives us the benefits of using results from complex analysis. For example, we use complex shifts and the residue theorem to help us derive the Parke-Taylor amplitude in the next section.

Now, for real momenta, we have the following relations,

$$[p|^a = (|p\rangle^{\dot{a}})^* \quad \text{and} \quad \langle p|_{\dot{a}} = (|p]_a)^* \tag{4.23}$$

$$\implies [pq]^* = \langle pq\rangle \tag{4.24}$$

$$\implies [k|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|k\rangle \tag{4.25}$$

4.4 Little Group Scaling

In the spinor helicity notation, we can see that the external data remains invariant under the transformations,

$$|p\rangle \longrightarrow t|p\rangle \quad \text{and} \quad |p] \longrightarrow t^{-1}|p] \tag{4.26}$$

where t is an arbitrary non-zero complex number for complex momenta p. For real momenta p, t has to be a complex phase, that is, t has to be of the form $e^{i\phi}$.

Recall that the little group is the subgroup of Lorentz (or Poincaré) group that leaves the momentum of the on-shell particle k^μ invariant. In the spinor helicity formalism little group scaling is given by the transformation in (4.26).

Little group scaling gives rise to a very important result,

$$\begin{aligned} &A_n[\{|1\rangle,|1],h_1\},...,\{t_i|i\rangle,t_i^{-1}|i],h_i\},...] \\ &\qquad = t_i^{-2h_i} A_n[\{|1\rangle,|1],h_1\},...,\{|i\rangle,|i],h_i\},...] \end{aligned} \tag{4.27}$$

This is the result that will help us determine the amplitudes of the 3-particle special kinematics.

4.5 3 Particle Special Kinematics

For a three particle scattering amplitude, momentum conservation implies

$$\sum_i p_i = 0 \tag{4.28}$$

Thus,

$$\begin{aligned} \langle 12\rangle[12] = 2p_1 \cdot p_2 = (p_1+p_2)^2 = (p_3)^2 = 0 \\ \implies \langle 12\rangle = 0 \quad \text{or} \quad [12] = 0 \end{aligned} \tag{4.29}$$

(If both of them are zero, then this implies that $p_1 = p_2 = p_3 = 0$) Say, we take $\langle 12\rangle \neq 0$ and $[12] = 0$.

$$\begin{aligned} \langle 12\rangle[23] &= -\langle 1|2|3] \\ &= \langle 1|1+3|3] \\ &= \langle 1|1|3] + \langle 1|3|3] \\ &= \langle 11\rangle[13] + \langle 13\rangle[33] \\ &= 0 \end{aligned} \tag{4.30}$$

where we have used equation (4.22) repeatedly and finally we have used the fact that $\langle pp\rangle = [pp] = 0$ due to the antisymmetric nature of the brackets. Thus we have $\langle 12\rangle[23] = 0$ and since we have $\langle 12\rangle \neq 0$, this implies that $[23] = 0$. Similarly, we can get $\langle 21\rangle[13] = 0$, and since $\langle 12\rangle \neq 0$, we have $\langle 21\rangle \neq 0$ and this implies that $[13] = 0$.

Therefore, when we take the case $\langle 12\rangle \neq 0$, we get

$$[12] = [23] = [13] = 0 \tag{4.31}$$

Now, from the fact that $[pq] = 0 \implies |p] \propto |q]$, we get,

$$|1] \propto |2] \propto |3] \tag{4.32}$$

Alternatively, if we considered the case that $\langle 12\rangle = 0$ and $[12] \neq 0$, then we would get,

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0 \tag{4.33}$$

$$\implies |1\rangle \propto |2\rangle \propto |3\rangle \tag{4.34}$$

Thus, for a 3 particle scattering of massless particles, the associated angle and square brackets must either be of the form (4.32) or (4.34). Thus a non-vanishing on-shell amplitude of 3 massless particles can only depend on square or angular brackets but never on both. Since for real momenta, the angle and square spinors are each other's conjugates, on-shell 3 particle amplitudes of massless particles can only be non-vanishing in complex momenta.

Thus, in the complex momentum case, we have two branches of solutions
1) $[12] = [23] = [13] = 0$
2) $\langle 12\rangle = \langle 23\rangle = \langle 13\rangle = 0$

Let us take the first branch to work with. We make a general ansatz

$$A_3(1^{h_1},2^{h_2},3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 13\rangle^{x_{13}} \tag{4.35}$$

where c is some constant. From little group scaling we get,

$$-2h_1 = x_{12}+x_{13} \quad -2h_2 = x_{12}+x_{23} \quad -2h_1 = x_{13}+x_{23} \tag{4.36}$$

On solving this system of equations, we get

$$A_3(1^{h_1},2^{h_2},3^{h_3}) = c\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 13\rangle^{h_1-h_2-h_3} \tag{4.37}$$

When we take $c = 1$ we get,

$$A_3(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 13\rangle} \tag{4.38}$$

When we reverse the helicities we obtain the solution from the other branch,

$$A_3(1^+,2^+,3^-) = \frac{[12]^3}{[23][31]} \tag{4.39}$$

4.6 MHV Amplitudes

Arguments from dimensional analysis can be used to prove that for $n > 3$ particle tree level amplitudes,

$$A_n[1^+,2^+,...n^+] = 0 \quad \text{and} \quad A_n[1^-,2^+,...n^+] = 0 \tag{4.40}$$

And, when we flip the helicities, we obtain,

$$A_n[1^-,2^-,...n^-] = 0 \quad \text{and} \quad A_n[1^+,2^-,...n^-] = 0 \tag{4.41}$$

Hence the only non-zero amplitudes are the ones with at least two helicities different from the helicity of the rest of the particles. The only exception to this rule is the $n = 3$ scattering amplitude, as we can see from equations (4.38) and (4.39) .

And the first of these non-zero amplitudes are the amplitudes with two negative helicity particles and $(n-2)$ positive helicity particles, which are called Maximally Helicity Violating (MHV) amplitudes.

The n-particle tree level MHV amplitude for $n \geq 3$ is given by the Parke-Taylor formula (We will derive this formula formally in the next section),

$$A_n[1^+,...,i^-,..j^-,...n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle...\langle n1\rangle} \tag{4.42}$$

And, the amplitudes with two positive helicity particles and $(n-2)$ negative helicity particles are called Anti-MHV amplitudes. For $n \geq 3$, the Anti-MHV n-particle tree level Parke-Taylor amplitude is given by,

$$A_n[1^+,...,i^-,..j^-,...n^+] = \frac{[ij]^4}{[12]...[n1]} \tag{4.43}$$

The $n = 3$ particle tree level amplitudes calculated in equations (4.38) and (4.39) are examples of the Parke–Taylor tree level MHV and Anti-MHV amplitudes given in equations (4.42) and (4.43) respectively.

5 Parke-Taylor Amplitude

5.1 Complex Shifts

An on-shell amplitude of n massless particle A_n is characterized by the momenta of the external particles and their helicities h_i's. They have two important properties :

$$p_i^2 = 0 \; \forall \; i = 1, 2, ...n. \qquad \text{and} \qquad \sum_{i=1}^{n} p_i^\mu = 0 \tag{5.1}$$

Let r_i^μ be n complex valued vectors, some of which may be 0, such that :

(i) $\sum_{i=1}^{n} r_i^\mu = 0$

(ii) $r_i \cdot r_j = 0 \; \forall \; i, j = 1, 2, ...n$. This $\Rightarrow r_i \cdot r_i = 0$

(iii) $p_i \cdot r_i = 0 \; \forall \; i = 1, 2, ...n$ (where the index i is not summed over).

These vectors are used to define the shifted momenta $\hat{p}_i^\mu$ as,

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \qquad \text{for some } \; z \in \mathbb{C} \tag{5.2}$$

Note that the shifted momenta $\hat{p}_i^\mu$ satisfy both the properties given in (5.1). Now we define, P_I^μ and R_I^μ as,

$$P_I^\mu = \sum_{i \in I} p_i^\mu \; ; \qquad R_I^\mu = \sum_{i \in I} r_i^\mu \tag{5.3}$$

and $\hat{P}_I^\mu$ is defined similarly. Now, we have,

$$\hat{P}_I^2 = P_I^2 + 2 \; P_I \cdot R_I$$

We see that $\hat{P}_I^\mu$ is linear in z. We now rewrite this equation as,

$$\Rightarrow \hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{where } z_I = -\frac{P_I^2}{2 \; P_I \cdot R_I} \tag{5.4}$$

We study the amplitude as a function of $\hat{p}_i^\mu$ instead of a function of p_i^μ, and call this the shifted amplitude. By construction, this shifted amplitude is a holomorphic function of z, given by $\hat{A}_n(z)$. The amplitude with the unshifted momenta A_n is obtained by setting $z = 0$, that is, $A_n = \hat{A}_n(z = 0)$.

We consider only the case where A_n is a tree level amplitude. We study the poles of the function $\hat{A}_n(z)$ in order to better understand the analytic structure of the function. The only way the shifted amplitude $\hat{A}_n(z)$ can have poles is from the shifted propagators $\frac{1}{\hat{P}_I^2}$. These poles occur when $\hat{P}_I^2 \longrightarrow 0$, and from equation (5.4), we can see that this happens at $z = z_I$. Now, poles of different propagators occur at different points. Hence, $\hat{A}_n(z)$ has only simple poles, all of which occur at $z \neq 0$.

Now, we recall some important results from complex analysis that we will be using as we move forward.
1) Cauchy's Integral Formula states that when D is a simply connected domain, bounded by a piecewise smooth curve γ, and f is an analytic function in a set U that contains the closure of D, then

$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - w} dz \quad \forall \; w \in D \tag{5.5}$$

2) Recall that the residue of a function f at a point z_0 is the value of the coefficient of $(z-z_0)^{-1}$ in the Laurent series expansion of the function in the neighbourhood of the point $z-z_0$.
3) Cauchy's Residue theorem that states that if γ is a simple, closed, positively oriented contour in the complex plane and $g(z)$ is analytic except for points $z_1, z_2, ...z_n$ inside the contour γ, then

$$\oint_\gamma g(z)dz \;=\; 2\pi i \sum_{k=1}^{n} Res_{z=z_k} g \tag{5.6}$$

We now consider the function $\hat{A}_n(z)$. Since, $A_n = \hat{A}_n(z=0)$, from equation (5.5) we have,

$$A_n = \frac{1}{2\pi i} \int_\gamma \frac{\hat{A}_n(z)}{z} dz \tag{5.7}$$

Now, lets define $g(z) = \dfrac{\hat{A}_n(z)}{z}$. Then, from (5.6), we have

$$\oint_\gamma \frac{\hat{A}_n(z)}{z} dz \;=\; 2\pi i \sum_{k=1}^{n} Res_{z=z_k} \frac{\hat{A}_n(z)}{z} \tag{5.8}$$

From equations (5.7) and (5.8), we get,

$$A_n = -\sum_{z_I} Res_{z=z_I} \frac{\hat{A}_n(z)}{z} \tag{5.9}$$

where the negative sign arises due to the orientation of the closed curve γ.

Now, at the z_I pole, the propogator $\dfrac{1}{\hat{P}_I^2} \longrightarrow 0$. Thus, in the limit $z \longrightarrow z_I$, the shifted amplitude factorises into two parts $\hat{A}_L(z_I)$ and $\hat{A}_R(z_I)$.

$$\begin{aligned} \text{as } \; z \longrightarrow z_I, \quad \hat{A}_n(z) \longrightarrow \;\; & \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) \\ & = -\frac{z_I}{z-z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \end{aligned} \tag{5.10}$$

Thus, as $z \longrightarrow z_I$, the residue of $\dfrac{\hat{A}_n(z)}{z}$ is given by $\hat{A}_L(z_I) \dfrac{1}{\hat{P}_I^2} \hat{A}_R(z_I)$. Now, since, A_n is the sum of the residues at all the poles z_I, we have

$$\hat{A}_n(z) = \sum_{diagrams I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) \tag{5.11}$$

where the sum is performed over all possible factorisation channels I, and over all possible on-shell particle states (helicity states and so on), that can be exchanged on the internal line.

Now, note that each unshifted amplitude factorises into shifted amplitudes of a smaller number of particles. Since, we already have the scattering amplitude for 3 particle scattering, we can calculate the amplitude for 4, 5 and any other higher finite number n particle scattering inductively. This is the principle behind the recursive relations used to calculate scattering amplitudes.

5.2 BCFW Recursion Relations

We define the BCFW shift as,

$$|\hat{i}] = |i] + z|j] \quad ; \quad |\hat{i}\rangle = |i\rangle \quad ; \quad |\hat{j}\rangle = |j\rangle - z|i\rangle \quad ; \quad |\hat{j}] = |j] \tag{5.12}$$

This is called an $[i, j\rangle$- shift. We can see that $[\hat{i}k]$ and $\langle\hat{j}k\rangle$ are linear in z for $k \neq i, j$, while the rest of the scalar products $[\hat{i}\hat{j}] = [ik]$, $[\hat{j}k] = [jk]$, $\langle\hat{i}\hat{j}\rangle = \langle ij\rangle$, and $\langle\hat{i}k\rangle = \langle ik\rangle$ remain unshifted.

The n-particle tree level MHV amplitude for $n > 3$ is given by the Parke-Taylor formula in equation (4.42). We prove this formula in this section using mathematical induction. We first consider a BCFW shift and then we assume the formula to be true for $n-1$ and prove it to be true for n.

Now, let's consider a $[1,2\rangle$- shift. (The same proof can be worked out with non-adjacent i and j too). We want to calculate $A_n[1^-, 2^-, 3^+, ...n^+]$ for the $[1,2\rangle$- shift.

$$\begin{aligned} A_n[1^-, 2^-, 3^+, ...n^+] = \sum_{k=4}^{n} \sum_{h_I = \pm} & A_{n-k+2+1}[1^-, \hat{P}_I^{h_I}, k^+, ...n^+] \frac{1}{P_I^2} \\ & \times A_{k-1}[-\hat{P}_I^{-h_I}, 2^-, 3^+ ...(k-1)^+] \end{aligned} \tag{5.13}$$

where we have labelled the internal momentum P_I and for $k = 4, 5, ...n$, we have used,

$$P_I = p_2 + p_3 + ... + p_{k-1} \quad ; \quad \hat{P}_I = \hat{p}_2 + p_3 + ... + p_{k-1} \tag{5.14}$$

$A[-++...+]$ amplitudes vanish except for $n = 3$. Thus, we have,

$$\begin{aligned} A_n[1^-, 2^-, 3^+, ...n^+] = & A_3[1^-, \hat{P}_{1n}^+, n^+] \frac{1}{P_{1n}^2} A_{n-1}[-\hat{P}_{1n}^-, 2^-, 3^+ ...(n-1)^+] \\ & + A_{n-1}[1^-, \hat{P}_{23}^-, 4^+, ...n^+] \frac{1}{P_{23}^2} A_3[-\hat{P}_{23}^+, 2^-, 3^] \end{aligned} \tag{5.15}$$

where we have used $P_{ij} = p_i + p_j$.

We first consider the first term in the RHS of equation (5.15). Let's consider the 3-particle amplitude in this term, $\hat{A}_3[1^-, \hat{P}_{1n}^+, n^+]$ and apply 3-particle special kinematics to it. This is an Anti-MHV amplitude. From our result in equation (4.39) we get,

$$\hat{A}_3[1^-, \hat{P}_{1n}^+, n^+] = \frac{[\hat{P}_{1n} n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]} \tag{5.16}$$

We use the following convention for analytic continuation,

$$|-p\rangle = -|p\rangle \quad \text{and} \quad |-p] = +|p] \tag{5.17}$$

Since we have $\hat{P}_{1n} = \hat{p}_1 + p_n$, we have

$$\hat{P}_{1n}^2 = 2\hat{p}_1 \cdot p_n = \langle\hat{1}n\rangle[\hat{1}n] = \langle 1n\rangle[\hat{1}n] \tag{5.18}$$

The $\hat{P}_{1n}$ going on-shell condition implies

$$\hat{P}_{1n}^2 = 0 = \langle 1n\rangle[\hat{1}n] \tag{5.19}$$

Therefore, for $\hat{P}_{1n}^2$ to go on-shell, z must take a value such that $[\hat{1}n] = 0$. Consider,

$$\begin{aligned} |\hat{P}_{1n}\rangle\,[\hat{P}_{1n}n] &= -\hat{P}_{1n}|n] \\ &= -(\hat{p}+p_n)|n] \\ &= -\hat{p}_1|n] \\ &= |\hat{1}\rangle\,[\hat{1}n] = 0 \end{aligned} \tag{5.20}$$

where, in the third line we have used the fact that $p_n|n] = 0$, and in the fourth line, we have used $[\hat{1}n] = 0$. Thus, we have

$$|\hat{P}_{1n}\rangle\,[\hat{P}_{1n}n] = 0 \tag{5.21}$$

Since $|\hat{P}_{1n}\rangle \neq 0$, we have $[\hat{P}_{1n}n] = 0$. Similarly, we can show that $[1\hat{P}_{1n}] = 0$. Therefore, all the spinor products in (5.16) vanish.

$$\implies \hat{A}_3[1^-, \hat{P}_{1n}^+, n^+] = 0 \tag{5.22}$$

Thus, the first term of equation (5.15) vanishes. In the second term, the 3-particle amplitude is $\hat{A}_3[-\hat{P}_{23}^+, \hat{2}^-, 3^+]$. This is an Anti-MHV amplitude too. In this case, it does not vanish because the shift of the momentum p_2 is on the angle spinor, and not on the square spinor.

$$\hat{A}_3[-\hat{P}_{23}^+, \hat{2}^-, 3^+] = \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]} \tag{5.23}$$

Now, from the Parke-Taylor formula for $(n-1)$, we have

$$A_{n-1}[1^-, \hat{P}_{23}^-, 4^+, ...n^+] = \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle\langle \hat{P}_{23}4\rangle\langle 45\rangle...\langle n\hat{1}\rangle} \tag{5.24}$$

When we put equations (5.22), (5.23) and (5.24) in (5.15), we get

$$A_n[1^-, 2^-, 3^+, ...n^+] = \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle\langle \hat{P}_{23}4\rangle\langle 45\rangle...\langle n\hat{1}\rangle} \frac{1}{\langle 23\rangle[23]} \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]} \tag{5.25}$$

Now,

$$\begin{aligned} \langle \hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] &= -\langle \hat{1}\hat{P}_{23}\rangle[\hat{P}_{23}3] \\ &= \langle \hat{1}|\hat{P}_{23}|3] \\ &= \langle \hat{1}|\hat{p}_2 + p_3|3] \\ &= \langle \hat{1}|\hat{p}_2|3] \\ &= -\langle \hat{1}\hat{2}\rangle[\hat{2}3] \\ &= -\langle 12\rangle[23] \end{aligned} \tag{5.26}$$

Similarly, we get,

$$\begin{aligned} \langle \hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] &= \langle 4|\hat{P}_{23}|2] \\ &= \langle 4|p_3|2] \\ &= -\langle 43\rangle[32] \end{aligned} \tag{5.27}$$

Thus, when we put equations (5.26) and (5.27) in (5.25), we get

$$\implies A_n[1^-,2^-,3^+,...n^+] = \frac{\langle 12\rangle^3[23]^3}{\langle 23\rangle[23]^3\langle 34\rangle\langle 45\rangle...\langle n1\rangle} \tag{5.28}$$

$$\implies A_n[1^-,2^-,3^+,...n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle...\langle n1\rangle} \tag{5.29}$$

Thus, we have derived the famous Parke-Taylor formula for $n > 3$ particle tree level MHV amplitudes.

6 Grassmanians

The Grassmannian is given by G(k,n),

$$G(k,n) = \{\text{Space of k-planes passing through the origin in n-dimensions}\}$$

$$\mathbb{P}^{n-1} = G(k=1,n)$$

That is, Projective space is a special case of Grassmannians.

An element of a Grassmannian, a k-plane in n-dim, is specified by k n-dim vectors which span the plane, given by a $k \times n$ matrix denoted by $\mathscr{C}_{\alpha\ a}$, where $\alpha = 1,2,...k$ and $a = 1,2,...n$.

The plane itself is invariant under a change of basis, that is, a GL(k) transformation, done by acting a $k \times k$ matrix, $\mathscr{L}^{\beta}_{\alpha}.\ \mathscr{C}_{\alpha\ a} \longrightarrow \mathscr{L}^{\beta}_{\alpha}\mathscr{C}_{\beta\ a}$.

Therefore the Grassmannian G(k,n) is the space of $k \times n$ matrices $\mathscr{C}$, modulo the space of $k \times k$ transformations GL(k), that is $G(k,n) = k \times n\ matrices/GL(k)$. Then,

$$dim[GL(k)] = (k \times n) - k^2 = k(n-k) \tag{6.1}$$

Given a k-plane, $\exists$ a unique (n-k) plane orthogonal to it in projective space. This is an extension of the notion of projective duality. Thus, we have $dim[G(n,k)] = dim[G(n-k,n)]$. This can be seen from equation (6.1).

Now, let's consider the scattering data of an n-particle scattering amplitude. We take the n spinors that determine the kinematical data of the scattering amplitude and group them into a matrix, given by $\Lambda = (|p_1]...|p_n])$. This is a $2 \times n$ matrix as the $|p_i]$'s are 2 dimensional. Thus we can think of Λ as a matrix made of n 2-dimensional coloumn vectors. Alternatively, we can think of Λ as a matrix made of 2 n-dimensional row vectors. Now, in this n-dim space, we have two vectors, $\vec{a}$ and $\vec{b}$, representing the data.

$$\Lambda = \begin{pmatrix} a_1 & a_2 & ... & a_n \\ b_1 & b_2 & ... & b_n \end{pmatrix} \tag{6.2}$$

An $SL(2)$ transformation acts on the $|p_i]$'s as $\vec{a'} = \alpha\vec{a} + \beta\vec{b}$ and $\vec{b'} = \gamma\vec{a} + \sigma\vec{b}$. Thus, we can see that under an $SL(2)$ transformation, the plane spanned by $\vec{a}$ and $\vec{b}$ remains invariant. Every new $\vec{a'}$ and $\vec{b'}$ belong to this plane.

Say, we want to write down our scattering data in a different Lorentz frame. Since the $|p_i]$'s are left handed Weyl spinors, a Lorentz transformation means we act on the $|p_i]$'s with an SU(2) matrix. Under this transformation, let $\vec{a} \longrightarrow \vec{a}'$ and $\vec{b} \longrightarrow \vec{b}'$. Since, $SU(2)$ is a subset of $SL(2)$, every new $\vec{a}'$ and $\vec{b}'$ belong to this plane. Hence the Lorentz invariant way of talking about this data of the $|p_i]$'s is not to specify the $|p_i]$'s individually, but to specify the 2-plane Λ in n-dim, given by $\Lambda^{2-plane}$. The different choice of Lorentz frames is the different choice of basis for the plane, but the plane itself remains invariant. And similarly, we can specify a 2-plane for the angle brackets, the $|p_i\rangle$'s too, given by $\tilde{\Lambda}$.

Therefore the kinematic data looks like this, with a $\Lambda^{2-plane}$ and a $\tilde{\Lambda}^{2-plane}$. From equation (4.16), momentum conservation gives $\sum_i |p_i\rangle [p_i| = 0$, which means that the dot product of a vector on the $\Lambda^{2-plane}$ with any vector on the $\tilde{\Lambda}^{2-plane}$ is 0. Thus, these planes are orthogonal to each other by momentum conservation. Therefore, $|p_i]$ is an n-vector on the $\Lambda^{2-plane}$ and $|p_i\rangle$ is an n-vector on the $\tilde{\Lambda}^{2-plane}$ [3].

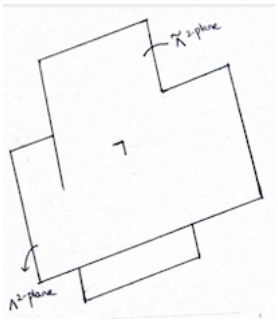

Figure 1: This planes $\Lambda^{2-plane}$ and $\tilde{\Lambda}^{2-plane}$ are orthogonal to each other.

The the kinematic space of n-particle scattering amplitudes is the space of two planes in the space of n-spinors, $\mathbb{C}^{2N}$, given by

$$G(2,2n) = \mathbb{C}^2 n/GL(2) \tag{6.3}$$

7 Momentum Twistor Formalism

The two 2-component objects, $|p]_a$ and $\langle p|_{\dot{b}}$ are called Spinor Helicity variables. Recall from equation (4.5) that $p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}}$. This equation makes the constraint $p^2 = 0$ manifest. But amplitudes are subject to another constraint, momentum conservation, given by $\sum_i |i\rangle [i| = 0$, which we have to impose on our Spinor Helicity variables. We thus define another set of variables, called momentum twistor variables, that makes this constraint manifest as well.

The spinor helicity variables $|p]_a$ and $\langle p|_{\dot{b}}$ are sometime written as λ_α and $\tilde{\lambda}_{i\dot{\beta}}$ respectively, where, like before, the dotted and undotted indices are the left and right handed $SU(2)$ indices respectively.

Now, consider the Minkowski spacetime but with a small caveat : each point does not denote a position in spacetime, it is the value of the variable X. We call this the X-Space. In this space, imagine placing all the null momenta, from

head to tail. Each p_i, is a null vector in Minkowski space. The last momentum p_n's head comes right onto the tail of the first momentum p_1, because the p_i's sum to 0.

An n-particle scattering amplitude has n null momenta. We define the X_i's to be

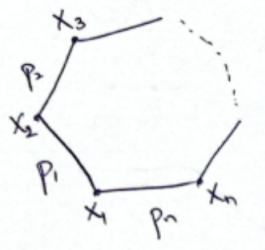

Figure 2: The polygon made of n null momenta.

$$p_i = X_{i+1} - X_i \tag{7.1}$$

You can start at any arbitrary point in the space, and you will end up back there. Note that this means, the X_i's are not uniquely determined, because we can arbitrarily translate in X-space.

Consider, $(p_i)_{\alpha\dot{\beta}} = (X_{i+1})_{\alpha\dot{\beta}} - (X_i)_{\alpha\dot{\beta}}$. We contract it on both sides with λ_i^{β}.

$$\varepsilon^{\alpha\beta}(p_i)_{\alpha\dot{\beta}}\ \lambda_{i\beta} = \varepsilon^{\alpha\beta}(X_{i+1} - X_i)_{\alpha\dot{\beta}}\ \lambda_{i\beta} \tag{7.2}$$

We write this equation as (this is the notational shorthand),

$$(p_i\ \lambda_i)_{\dot{\beta}} = (X_{i+1}\ \lambda_i)_{\dot{\beta}} - (X_i\ \lambda_i)_{\dot{\beta}} \tag{7.3}$$

Now, since $\varepsilon^{\alpha\beta}$ is an anti-symmetric tensor, we have $\varepsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{i\beta} = 0$.

$$\begin{aligned} \Rightarrow\ \varepsilon^{\alpha\beta}(p_i)_{\alpha\dot{\beta}}\ \lambda_{i\beta} &= \varepsilon^{\alpha\beta}(\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\beta}})\ \lambda_{i\beta} \\ &= \varepsilon^{\alpha\beta}(\lambda_{i\alpha}\lambda_{i\beta})\ \tilde{\lambda}_{i\dot{\beta}} = 0 \end{aligned} \tag{7.4}$$

$$\Rightarrow (X_{i+1}\ \lambda_i)_{\dot{\beta}} - (X_i\ \lambda_i)_{\dot{\beta}} = 0 \tag{7.5}$$

We now define $\mu_i^{\dot{\alpha}} = (X_{i+1}\lambda_i)^{\dot{\alpha}} = (X_i\lambda_i)^{\dot{\alpha}}$ where $(X_{i+1}\lambda_i)^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}(X_{i+1}\lambda_i)_{\dot{\beta}}$. We now have 2 λ's and 2 μ's. We combine these into a single four component object, Z_i^I, defined as,

$$Z_i^I = (\lambda_i^1,\ \lambda_i^2,\ \mu_i^{\dot{1}},\ \mu_i^{\dot{2}}) \tag{7.6}$$

where the I index stands for the α index for the λ's and for the $\dot{\alpha}$ index for the μ's [14].

Recall that the little group scaling gives

$$(\lambda,\ \tilde{\lambda}) \longrightarrow (t^{-1}\lambda,\ t\tilde{\lambda}) \tag{7.7}$$

This transformation keeps the p_i's, and hence X_i's, constant. Note that μ also transforms similar to λ. Hence, all four components of Z_i^I transform homogeneously,

$$Z_i^I \longrightarrow tZ_i^I \tag{7.8}$$

Therefore, Z_i^I is properly understood as a point in projective space $\mathbb{P}^3$. The projective space is complex for complex momenta.

We define,

$$\langle Z_a\, Z_b\, Z_c\, Z_d \rangle = \varepsilon^{IJKL} Z_{aI} Z_{bJ} Z_{cK} Z_{dL} \tag{7.9}$$

8 The Amplituhedron

8.1 Positive Geometries and Canonical Forms

A positive geometry is a region in D-dimensional space, cut out by a finite sequence of polynomials/inequalities that satisfies the following two conditions :

1) Every boundary of the D-dimensional positive geometry is a (D-1)-dimensional positive geometry itself, i.e, the positive geometry has boundaries of all co-dimensions.

2) There exists a unique, non-zero, rational D-form (D-dimensional differential form) defined on the positive geometry, such that on taking the residue of the D-form on a boundary of the positive geometry yields the unique (D-1) form defined on the boundary. This differential form is known as the canonical form associated with the positive geometry.

Now consider a square. It is a 2-d object. It has both edges and vertices, that is, boundaries of both 1 and 0 dimensions. (Boundaries of all co-dimensions is a necessary condition). By this definition, a circle is not a positive geometry.

Examples

Consider an interval (a,b) on the z-axis. It is a 1-dim positive geometry. Its canonical form is given by,

$$\Omega = \frac{dz}{z-a} - \frac{dz}{z-b} \tag{8.1}$$

Consider a triangle, cut out by the equations, $x = 0$, $y = 0$ and $1 - x - y = 0$. Its canonical form is given by,

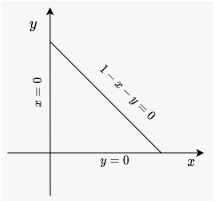

Figure 3: The triangle as a positive geometry

$$\Omega = \frac{dxdy}{xy(1-x-y)} \tag{8.2}$$

Taking its residue on the $x = 0$ boundary, the interval (0,1) on the y axis, we get,

$$Res\ \Omega_{|_{x=0}} = \frac{dy}{y(1-y)} = \frac{dy}{y} - \frac{dy}{1-y} \tag{8.3}$$

Given a positive geometry, there are a few methods to determine the canonical form associated with it :

1)Triangulation : Consider a quadrilateral.

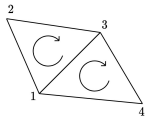

Figure 4: Triangulation of a parallellogram

The canonical form for a quadrilateral is equal to the sum

$$\Omega^{quad} = \Omega^{123} + \Omega^{134} \tag{8.4}$$

Note that the singularities lying on the line 13 get cancelled when we add both triangles due to orientation. Thus we have to keep orientation in mind when performing this sum. This method is called triangulation.

If a positive geometry $\mathscr{P}$ can be tiled with smaller positive geometries $\mathscr{P}_i$ in a way such that they are non overlapping, as shown in the figure,

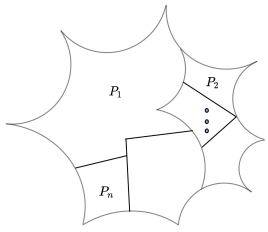

Figure 5: Triangulation of a positive geometry $\mathscr{P}$

then its canonical form is given by,

$$\Omega^{\mathscr{P}}_{cann} = \sum_i \Omega^{\mathscr{P}_i}_{cann} \tag{8.5}$$

2)Push Forward : We consider a positive geometry P_1, whose canonical form we know, like a triangle. We can define a diffeomorphism ϕ from P_1 to a positive geometry P_2, whose canonical form we are looking for. Then the canonical form of P_1 can be pushed forward via this map ϕ to the canonical form of P_2 [2].

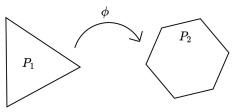

Figure 6: The push forward

Some other methods to determine the canonical form of a given positive geometry include :

3) "Volume" as Canonical Form : [6]

4) Stringy Canonical Forms : [4]

8.2 $\mathcal{N} = 4$ SYM

We use the momentum twistor variables to parameterise the kinematic space of $\mathcal{N} = 4$ SYM. For an n particle scattering amplitude, the external data is given by the n Z_i^I's. This is a 4n dimensional space.

In a loop level Feynman diagram, the loop variables are represented by lines in twistor space. A line in twistor space is denoted by (AB) where this is the line joining the points Z_A and Z_B. Thus, to specify a loop we have to specify 2 points in twistor space. Thus, to specify a loop variable, we have 8 degrees of freedom. But, we have (Z_A, Z_B) upto a linear transformation $(\alpha Z_A + \beta Z_B, \rho Z_A + \sigma Z_B)$, because $\alpha Z_A + \beta Z_B$ leads to another point on the same line. Thus, there is a GL(2) action on the points Z_A and Z_B. This GL(2) action reduces the degrees of freedom needed to specify the line denoting the loop variable to 4. Thus, in an L loop diagram, the loop variables are parametrised by a 4L dimensional space.

Thus, the kinematic space of an L loop diagram of a n particle scattering amplitude is parametrised by the n Z's, and the L lines in projective space, that is, by Z_i where $i = 1, 2, ...n$ and by $(AB)_\alpha$ where $\alpha = 1, 2, ...L$. This is a $4(n+L)$ dimensional space.

An L loop scattering amplitude is given by

$$\mathcal{M}_{n,k}[Z_a^I, \eta_a^I, d^4(AB)_\alpha] \tag{8.6}$$

where $a = 1, 2, 3...n$, $\alpha = 1, 2, 3...L$, k is the number of negative helicity gluons in the scatering amplitude and the η_a^I's are the grassmann variables that are the eigenvalues that we get by diagonalising Q^I, the operator from the Supersymmetry algebra.

And we obtain a differential form from these scattering amplitudes by making the repleacement $\eta_a^I \longrightarrow dZ_a^I$,

$$\mathcal{M}_{n,k}[Z_a^I, \eta_a^I, d^4(AB)_\alpha] \longrightarrow \Omega_{4(k+L)}[Z_a^I, dZ_a^I, d^4(AB)_\alpha] \tag{8.7}$$

Now we define a positive geometry called the Amplituhedron, whose associated canonical form is exactly the differential form obtained above.

On our kinematic space, we impose the following positivity conditions to define our positive geometry.

1. We require that the sequence
$$\{\langle Z_b, Z_{b+1}, Z_a, Z_i\rangle \mid i = 1,2,...n \ (except\ a)\} \tag{8.8}$$
has k sign flips.

2. We require that the sequence
$$\{\langle AB_\alpha, Z_a, Z_i\rangle \mid i = 1,2,...n \ (except\ a)\} \tag{8.9}$$
has k+2 sign flips.

3. Also, the L lines must not intersect. We ensure this by requiring that these brackets are positive,
$$\langle AB_\alpha AB_\beta\rangle > 0 \tag{8.10}$$

Imposing these positivity conditions gives us a positive geometry with a unique $4(n+L)$ dimensional canonical form associalted with it.

But we need a $4(k+L)$ dimensional differential form for our scattering amplitude.

We define a 4k dimensional subspace in the 4n dimensional space of the Z's. (effectively, we define a $4(k+L)$ dimensional subspace in the $4(n+L)$ dimensional space). This subspace is defined by fixing a point Z^I_{*a} and translating it in 4k directions.

$$\begin{aligned}\mathscr{S}_{4k} &= \{Z^I_a \mid Z^I_a = Z^I_{*a} + y^I_\alpha \Delta^\alpha_a\} \\ &= \text{ 4k subspace of Z-space}\end{aligned} \tag{8.11}$$

where $\alpha = 1,2,..k$ and the Δ^α_a's are fixed.

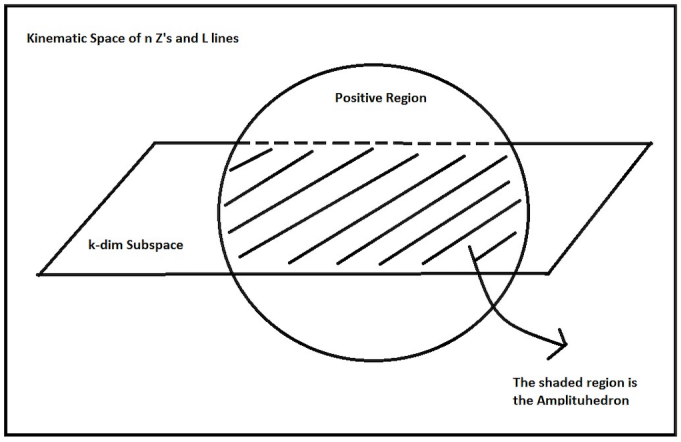

Figure 7: Pictorial representation of the Amplituhedron

We define the intersection of this subspace with our positive region to be our Amplituhedron. Our Amplituhedron is now a $4(k+L)$ dimensional positive geometry and its canonical form is exactly the differential form we obtained from our scattering amplitude by making the replacement $\eta_a^I \longrightarrow dZ_a^I$. [5]

Recall that when one the of virtual particles goes on shell, the amplitude blows up and on taking the residue of that amplitude, we get the factorisation of the amplitude into the two smaller amplitudes. On the Amplituhedron, one of the virtual particles going on shell can be directly related to going to the boundary of the Amplituhedron, where the canonical form blows up and on taking the residue, you obtain the canonical form of the dimension of the boundary that you are on. It was this property that first prompted Professor Arkani-Hamed and his team to look at positive geometries as a possible alternative to calculating scattering amplitudes.

8.3 Scalar Bi-Adjoint ϕ^3 Theory

This method is not specific to supersymmetric theories. In fact, such a formalism has been developed for the Scalar Bi-Adjoint ϕ^3 Theory. In the case of the ϕ^3_{BA} theory, the variables we use to parametrise the Kinematic space are the Mandelstam Invariants, and the positive geometry we obtain in this case is called the Associahedron. This Associahedron also has the property that on its boundaries it factorises in Associahedra of lower dimensions [1].

Part II

Loop Quantum Gravity

Loop Quantum Gravity (LQG) is an attempt to obtain a non-perturbative background independent formulation of gravity in 4 and higher dimensions.

9 The Hamiltonian Approach

The action for gravity and matter is given by

$$S_{GR}[g_{\mu\nu}] = S_{EH+\Lambda} + S_M \tag{9.1}$$

$$S_{GR}[g_{\mu\nu}] = \int d^4x\sqrt{-g}\,\left\{\frac{1}{\kappa}(R-\Lambda) + \mathscr{L}_M\right\} \tag{9.2}$$

where $\kappa = 16\pi G_N$. Variation of the action gives the Einstein Field Equations,

$$\frac{\delta S_{GR}}{\delta g_{\mu\nu}} = 0 \implies G_{\mu\nu} = 8\pi G_{\mu\nu}T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{9.3}$$

$$\text{where } G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \text{ and } T_{\mu\nu} = \frac{\delta S_M}{\delta g_{\mu\nu}} \tag{9.4}$$

The action S_{GR} is more fundamental than the Einstein field equations. Matter equations of motion arise from energy momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{9.5}$$

9.1 ADM Formalism

Given a hyper surface it is classified as null, spatial or time like. If its normal vector is n^μ, then the classification goes as

$$n_\mu n^\mu = \begin{cases} -1 & \text{spacelike hypersurface} \\ +1 & \text{timelike hypersurface} \\ 0 & \text{null hypersurface} \end{cases} \tag{9.6}$$

Let M be our spacetime manifold. It is a four dimensional manifold. We can foliate it as

$$M \cong \Sigma(t) \times \mathbb{R} \tag{9.7}$$

We then define a vector field t^μ, called the time evolution vector field. Since we are in a curved background in general the time evolution vector field need not be orthogonal to the spacial hypersurfaces (The hypersurfaces can have extrinsic curvature). Hence, if we decompose this vector at a point into two vectors tangential and orthogonal to the hypersurface at that point, we get the decomposition,

$$t^\mu = Nn^\mu + N^\mu \tag{9.8}$$

where N is a function called the Lapse function, $N : M \longrightarrow \mathbb{R}$, N^μ is a vector field called the shift vector field and n^μ is the unit normal vector to the hypersurface. The metric $g_{\mu\nu}$ exists independently of how you chose to foliate your space time.

Now, we define

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{9.9}$$

Thus, for any vector v^μ,

$$h_{\mu\nu}v^\nu = v_\mu + n_\mu n_\nu v^\nu$$

Now, if $v^\mu = n^\mu$, then $h_{\mu\nu}n^\nu = 0$ (because our hypersurfaces are spacelike, which implies that $n_\mu n^\mu = -1$). This means that h contracted with any vector perpendicular to the hypersurface is 0. This implies h lies completely on the hyper surface. It has only spatial components.

Now, you can always take a lower rank tensor and write it as a higher rank tensor by adding extra rows and columns of zeroes. This requires making a certain choice of reference frame but you can always make such a choice. Thus we can write $h_{\mu\nu}$ as a purely spatial tensor, i.e one whose non-spatial components are zero. Thus, we can write this tensor as h_{ab} where $\{e_a\}$ are the basis vectors that span the hypersurface for $a = 1, 2, 3$.

Since a tensor is a coordinate independent object, you may make a coordinate choice where your tensor has non 0 components in the extra rows too, but even in this choice its contraction with the perpendicular vector (whose components are again in the new choice) is still 0.

h_{ab} depends on the choice of the spatial surface. Thus the Hamiltonian formulation of General Relativity is foliation dependant. So, in some sense this formulation breaks general covariance.

We define a derivative operator as,

$$D_a\phi = h_a^\mu \nabla_\mu \phi \tag{9.10}$$

where h_a^μ is the 3×4 matrix produced by removing the row of zero elements from $h_{\mu\nu}$. Note that this is an abuse of notation, but is done to avoid confusion between a generic $h_{\mu\nu}$ and one in a specific choice of reference frame where its spatial components are zero. We have a 3-metric h_{ab} on our hypersurface $\Sigma(t)$. The covariant derivative D_a, defined in (9.10) turns out to be the covariant derivative compatible with h_{ab}. Its action on a tensor with covariant and contravariant indices is given by

$$D_a T_c^a = h_a^\mu h_\nu^b h_c^\alpha (\nabla_\mu T_\alpha^\nu) \tag{9.11}$$

9.2 Extrinsic Curvature

A 2-dim plane has no intrinsic curvature, but if it is bent into a cylinder and embedded in 3-dim flat space, it has extrinsic curvature. Another way to think about this difference is - Intrinsic curvature is the curvature measured by an observer on the hypersurface itself while extrinsic curvature is the curvature measured an external observer.

We define Extrinsic curvature $K_{\alpha\beta}$ as

$$K_{\alpha\beta} = h^{\mu}_{\alpha} h^{\nu}_{\beta} \nabla_{\mu} n_{\nu} \tag{9.12}$$

or equivalently,

$$K_{ab} = h^{\mu}_{a} h^{\nu}_{b} \nabla_{\mu} n_{\nu} \tag{9.13}$$

By definition, K_{ab} is symmetric. We can show that

$$2K_{\alpha\beta} = \mathscr{L}_{\vec{n}} h_{\alpha\beta} \tag{9.14}$$

We have,

$$D_a K^a_b - D_b K^a_a = R_{cd} n^d h^c_b \tag{9.15}$$

Now the spatial curvature tensor is given by,

$${}^{(3)}R_{abc}{}^{d} w_d = (D_a D_b - D_b D_a) w_c \tag{9.16}$$

We write the 3-curvature tensor ${}^{(3)}R_{abc}{}^{d}$ in terms of the 4-curvature tensor ${}^{(4)}R_{\mu\nu\alpha}{}^{\beta} = {}^{(4)}R_{abc}{}^{d}$ using the Gauss-Codazzi equation :

$${}^{(4)}R_{\mu\nu\alpha}{}^{\beta} = {}^{(3)}R_{abc}{}^{d} + K_{ac}K^d_b - K_{bc}K^d_a \tag{9.17}$$

$$\implies {}^{(4)}R = {}^{(3)}R + K^2 - K_{ab}K^{ab} \tag{9.18}$$

9.3 The Hamiltonian

From equation (9.1) we have,

$$g_{\mu\nu} = h_{\mu\nu} - n_{\mu} n_{\nu} \tag{9.19}$$

From this we get,

$$\sqrt{-g} = N\sqrt{h} \tag{9.20}$$

Consider the Einstein-Hilbert action

$$S_{EH} = \frac{1}{\kappa} \int d^4x \sqrt{-g} \;\; {}^{(4)}R \tag{9.21}$$

On putting equations (9.18) and (9.20) in the previous equation, we obtain,

$$S_{EH} = \frac{1}{\kappa} \int d^4x \, N\sqrt{h} \, [\, {}^{(3)}R + K^2 - K_{ab}K^{ab}] \tag{9.22}$$

We now evaluate the conjugate momentum of our configuration variable h_{ab}.

$$\Pi^{ab} = \frac{\delta \mathscr{L}_{EH}}{\delta \dot{h}_{ab}} \tag{9.23}$$

$$\Pi^{ab} = \sqrt{h}(K^{ab} - K h^{ab}) \tag{9.24}$$

Π^{ab} is a function of the extrinsic curvature because the extrinsic curvature encodes how the intrinsic metric changes.

Hence our phase space variables are $\{h_{ab}, \Pi^{ab}\}$. Now we define the Hamiltonian density. The Hamiltonian density is given by a Legendre transform of of the Lagrangian density.

$$\mathscr{H}_{EH} = \Pi^{ab}\dot{h}_{ab} - \mathscr{L}_{EH}$$

$$\mathscr{H}_{EH} = \sqrt{h}\left\{N\left(-{}^{(3)}R + \frac{1}{h}\Pi_{ab}\Pi^{ab} - \frac{1}{2h}\Pi^2\right) - 2N_b\left(D_a\left[\frac{1}{\sqrt{h}}\Pi^{ab}\right]\right)\right\} \tag{9.25}$$

where we have made the assumption that our spacetime is asymptotically flat and thus ignored the boundary term. Now, our equations of motion are given by the Hamilton's equations

$$\dot{h}_{ab} = \frac{\delta H_{EH}}{\delta \Pi^{ab}} \quad ; \quad \dot{\Pi}^{ab} = -\frac{\delta H_{EH}}{\delta h_{ab}}$$

Note that the Lagrangian does not depend on the derivatives of the shift and lapse vectors. The shift and lapse vectors behave like lagrange multipliers. Thus we have two constraint equations – The Hamiltonian constraint and the Diffeomorphism constraint given by equations (9.26) and (9.27) respectively:

$$\left(-{}^{(3)}R + \frac{1}{h}\Pi_{ab}\Pi^{ab} - \frac{1}{2h}\Pi^2\right) = 0 \tag{9.26}$$

$$D_a\left(\frac{1}{\sqrt{h}}\Pi^{ab}\right) = 0 \tag{9.27}$$

The Poisson brackets of our phase space variables are given by,

$$\{\Pi^{ab}(x), h_{cd}(x')\} = \delta^a_{(c}\delta^b_{d)}\delta(x-x') \tag{9.28}$$

10 Quantization

As our first step in quantization, we convert out variables into operators and then convert Poisson brackets into commutators,

$$\{h_{ab}, \Pi^{ab}\} \longrightarrow \{\hat{h}_{ab}, \hat{\Pi}^{ab}\}$$

$$[\hat{\Pi}^{ab}(x), \hat{h}_{cd}(x')] = i\hbar\ \delta^a_{(c}\delta^b_{d)}\delta(x-x') \tag{10.1}$$

$$[\hat{h}_{ab}(x), \hat{h}_{cd}(x')] = 0 \quad ; \quad [\hat{\Pi}^{ab}(x), \hat{\Pi}^{cd}(x')] = 0 \tag{10.2}$$

$$\hat{\Pi}^{ab}(x) = -i\hbar\ \frac{\delta}{\delta h_{ab}(x)} \tag{10.3}$$

The configuration variable is h_{ab}. The configuration space is the space of 3-metrics $\mathscr{C}$. States are functionals of the configuration space. The states map the configuration space $\mathscr{C}$ to the space of complex numbers, $\psi : \mathscr{C} \longrightarrow \mathbb{C}$, $\psi = \psi(h_{ab})$. In order to obtain the physical states of the theory ψ_p, we have to impose the constraint equations. The physical states of the system will be the ones that satisfy

$$\hat{\mathscr{H}_1}\psi_p(h_{ab}) = 0 \quad ; \quad \hat{\mathscr{H}_2}\psi_p(h_{ab}) = 0 \tag{10.4}$$

where $\hat{\mathscr{H}_1}$ and $\hat{\mathscr{H}_1}$ are the Hamiltonian and Diffeomorphism constraint respectively. The physical states of the system are the ones annihilated by the constraints.

Generally the Hamiltonian density is a polynomial function of the configuration and the momentum variables. But in the case of General Relativity, the Hamiltonian density is non polynomial in configuration and momentum variables.

11 Tetrad Formalism

11.1 Introduction

A tetrad is a choice of a local Lorentz frame. We have a set of 4-vectors defined at every point in the spacetime manifold, given by $e^I_\mu(x)$ where $I = 0,1,2,3$. These 4-vectors satisfy the condition,

$$e^I_\mu(x) e^J_\nu(x) \eta_{IJ} = g_{\mu\nu}(x) \tag{11.1}$$

The tetrad is also known as a soldering form. Since specifying the tetrads at each point is akin to specifying the metric tensor at each point, the tetrads become our new configuration variable.

Now that we have our configuration variable, the tetrads, we need to know how the tetrads change from one point to another, i.e we need to define a connection variable.

A connection tells us how to parallel transport vectors, tensors and other such objects from one point on the manifold to another. It does this by allowing us to define the covariant derivative, which measures the infinitesimal change in any tensorial object as you go from one point to another. This is why it is called a connection, since it connects tangent spaces from one point to another.

The symmetry group of Minkowski space is SO(3,1). But SO(3,1) and SL(2, $\mathbb{C}$) have the same lie algebra, hence we define an SL(2, $\mathbb{C}$) spin connection, since SL(2, $\mathbb{C}$) has the added advantage of being able to represent spinors too. Currently we are looking at a pure gravitational theory, but using an SL(2, $\mathbb{C}$) spin connection comes in handy when we add matter to the theory.

We define an SL(2,C) spin connection ω_μ and use two anti-symmetric indices to denote it, instead of one index. It is a lie algebra valued one-form.

$$\omega_\mu(x) = \omega^{IJ}_\mu(x) \mathscr{T}_{IJ} \tag{11.2}$$

where $\mathscr{T}_{IJ}$ are the generators of the SL(2, $\mathbb{C}$) lie group.

Hence, in the tetrad, the μ and ν are called the spacetime indices and the I and J are the SL(2, $\mathbb{C}$) indices, also called internal indices.

Ultimately, what happens is that when we write GR as a gauge field theory, then our tetrad becomes a dynamical variable and the SL(2, $\mathbb{C}$) spin connection ω^{IJ}_μ, becomes the corresponding gauge connection, that parallel transports this tetrad from point to point. The Einstien-Hilbert action can be written in a form which involves the curvature tensor of this gauge field, multiplied by one or more copies of the tetrad. The point of doing all this is, we can use our methods of quantisation of gauge theory to quantise gravity in these variables, the tetrad and the gauge connection $(e^I_\mu(x), \omega^{IJ}_\mu(x))$.

Thus, now we define a covariant derivative with respect to the spin connection ω_μ, acting on an object with both

spacetime and internal indices as,

$$\mathscr{D}_\mu e^I_\nu = \partial_\mu e^I_\nu - \Gamma^\rho_{\mu\nu} e^I_\rho + \omega^I_{\mu\ J} e^J_\nu \tag{11.3}$$

where Γ is the Christoffel connection associated with the metric $g_{\mu\nu}$. From the previous equation, we get, the action of the gauge covariant derivative on an object with only spacetime or only internal indices,

$$\mathscr{D}_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho \tag{11.4}$$

$$\mathscr{D}_\mu v^I = \partial_\mu v^I + \omega^I_{\mu\ J} v^J \tag{11.5}$$

Now, recall the metric compatibility condition in the metric formalism.

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{11.6}$$

We have a similar condition in the Tetrad formalism.

$$\mathscr{D}_\mu e^I_\nu = 0 \tag{11.7}$$

This is called the tetrad compatibility condition. This is the condition that the tetrad is parallel transported by the gauge covariant derivative.

Now, in equation (11.3), the Christoffel symbol Γ and the SL(2, $\mathbb{C}$) spin connection ω^{IJ}_μ appear to be independent, but when we impose the tetrad compatibility condition, we can solve for Γ and ω^{IJ}_μ, purely in terms of the tetrad. Similar to the metric formalism, where the metric compatibility condition relates the Christoffel symbol to the metric, the tetrad compatibility condition relates the SL(2, $\mathbb{C}$) spin connection ω^{IJ}_μ to the tetrads in the tetrad formalism.

$$\omega^{IJ}_\mu = \frac{1}{2} e^{\rho[I} \left(\partial_{[\alpha} e^{J]}_{\rho]} + e^{|\beta|J]} e^K_\alpha \partial_\beta e_{\rho K} \right) \tag{11.8}$$

Now the action of the commutator of the gauge covariant derivative on an object that has only spacetime indices is just our usual Riemann curvature tensor.

$$[\mathscr{D}_\mu, \mathscr{D}_\nu] v_\alpha = [\nabla_\mu, \nabla_\nu] v_\alpha = R^\beta_{\mu\nu\alpha} v_\beta \tag{11.9}$$

We can also define this commutator to be acting on an object with internal indices.

$$[\mathscr{D}_\mu, \mathscr{D}_\nu] v_I = R_{\mu\nu I}^{\ \ \ J} v_J \tag{11.10}$$

Recall that the tetrad is just a choice of Lorentz frame, a chage of basis. This means, any object with internal indices can be rewritten in terms of an object with spacetime indices as follows :

$$v_\alpha = e^I_\alpha v_I \tag{11.11}$$

On using the previous equation in (11.9), we get,

$$R^\beta_{\mu\nu\alpha} = e^I_\alpha e^\beta_J R_{\mu\nu I}^{\ \ \ J} \tag{11.12}$$

This is the key expression that underlies the equivalence between the tetrad formulation and the metric formulation of gravity. On contracting on both sides of this equation, we get,

$$R = R_{\mu\nu}{}^{IJ} e_I^\mu e_J^\nu \tag{11.13}$$

Now, on taking determinant on both sides of this equation (11.1), we get,

$$\sqrt{-g} = det(e) \tag{11.14}$$

Recall from equation (9.21) that the Einstein Hilbert action in the metric formalism is

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R \tag{11.15}$$

From equations (11.13) and (11.14), the Einstein Hilbert action in the tetrad formalism becomes,

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\, det(e) R_{\mu\nu}{}^{IJ} e_I^\mu e_J^\nu \tag{11.16}$$

This action is called the Palatini action and can be written as

$$S_p[e,\omega] = \frac{1}{2\kappa}\int d^4x Tr[(e^I \wedge e^J)^* \wedge R^{KL}\varepsilon_{IJKL}] \tag{11.17}$$

where $(e^I \wedge e^J)^*$ is the dual of $(e^I \wedge e^J)$. Thus,

$$S_p[e,\omega] = \frac{1}{4\kappa}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta}\ \varepsilon_{IJKL}\, e_\mu^I e_\nu^J R_{\alpha\beta}{}^{KL} \tag{11.18}$$

11.2 $(3+1)$ Splitting

We perform the $(3+1)$ splitting in the tetrad formalism. The Minkowski space is foliated by a family of 3-dimensional Euclidean surfaces.

$$\mathbb{R}^{3,1} \longrightarrow \mathbb{R}^3 \tag{11.19}$$

Recall that $SO(3,1)$ is the symmetry group of the Minkowski space, and that we have taken an $SL(2,\mathbb{C})$ spin connection on this space because $SL(2,\mathbb{C})$ and $SO(3,1)$ have the same lie algebra. Similarly, $SO(3)$ is the symmetry group of Euclidean 3-dim space, but since $SO(3)$ and $SU(2)$ have the same lie algebra, we define an $SU(2)$ spin connection on the hypersurface.

$$SL(2,\mathbb{C}) \longrightarrow SU(2) \tag{11.20}$$

$$\omega_\mu^{IJ} \longrightarrow \omega_a^{ij} \tag{11.21}$$

To perform the foliation, we introduce the lapse funcion N, the shift vector N^μ, the unit normal vector n^μ and the time evolution vector field t^μ. The tetrad then splits into a triad

$$e_I^\mu \longrightarrow e_i^a \tag{11.22}$$

where $a = 1,2,3$ are the spatial indices and $i = 1,2,3$ are the internal indices, the $SU(2)$ indices. Now, the tetrad equation in (11.1) splits as,

$$g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ} \longrightarrow h_{ab} = e_a^i e_b^i \delta_{ij} \tag{11.23}$$

since η is the metric of Minkowski space and δ is the metric of Euclidean 3-d space.

Recall that, in the ADM formalism, the phase space variables are h_{ab} and Π^{ab}. But since Π^{ab} can be written completely in terms of K_{ab}, alternatively, we can consider h_{ab} and K_{ab} to be our phase space variables.

To keep on par with our (3+1) splitting, we define a new variable, instead of K_{ab}, given by,

$$K_a^i = K_{ab}e^{bi} = K_a^b e_b^i \tag{11.24}$$

We can invert the previous equation, to obtain,

$$K_{ab} = e_{(a}^i K_{b)}^j \delta_{ij} \tag{11.25}$$

Since, K_{ab} is symmetric

$$K_{[ab]} = 0 \implies e_{[a}^i K_{b]}^j \delta_{ij} = 0 \tag{11.26}$$

The last equation in (11.26) is known as the Gauss constraint.

11.3 Ashtekar Variables

The Hamiltonian density given in (9.25) is non polynomial in configuration and momentum variables. The non polynomial nature of the constraints and by extension the Hamiltonian makes it impossible to solve them. We now make yet another change of variable to a new set of variables called the Ashtekar variables. Rewriting the Hamiltonian in terms of the Ashtekar variables gives you polynomial constraints which turn out to be exactly solvable and when you solve them, the states can be expressed in terms of certain graphs (arbitrary graphs whose edges are labelled by spin) called spin networks. On these spin networks we can define area and volume operators. The spin networks are completely background independent. In fact geometry emerges from these spin network states.

We define a new variable $\widetilde{E}_i^a$, called the densitised triad, as,

$$\widetilde{E}_i^a = \frac{1}{2i}\varepsilon^{abc}\varepsilon_{ijk}e_b^j e_c^k \tag{11.27}$$

We also define the SU(2) spin connection in terms of one index, instead of two anti-symmetric indices (this can be done in 3 dimensions)

$$\Gamma_a^i = \varepsilon_{jk}^i \omega_a^{jk} \tag{11.28}$$

And, now, we define our new connection variable A_a^j,

$$A_a^j = \Gamma_a^j - iK_a^j \tag{11.29}$$

The Ashtekar variables are $\widetilde{E}_a^j$ and A_a^i.

The Christoffel symbol tells us how to parallel transport vectors, but it is not a true connection in the sense of being a section of a fibre bundle. But A_a^i is a true connection.

When we consider $\widetilde{\mathscr{D}}_a$ to be the covariant derivative with respect to the connection variable A_a^i, the metric compatibility condition becomes

$$\widetilde{\mathscr{D}}_a e_b^i = 0 \tag{11.30}$$

Now, in terms of the Ashtekar variables $\widetilde{E}_a^j$ and A_a^i, the constraints become,

$$\widetilde{\mathscr{D}}_a \widetilde{E}_i^a = 0 \tag{11.31}$$

$$\varepsilon_k^{ij} \widetilde{E}_i^a \widetilde{E}_j^b F_{ab}^k = 0 \tag{11.32}$$

$$\widetilde{E}_i^b F_{ab}^i - A_a^i \widetilde{\mathscr{D}}_b \widetilde{E}_i^b = 0 \tag{11.33}$$

where (11.31) is the Gauss constraint, (11.32) is the Hamiltonian constraint and (11.33) is the Diffeomorphism constraint.

12 Introduction to Lattice Variables

Let Σ be a 3-d surface. We put an integer lattice γ on it. This lattice has been shown in blue in the figure below. Let the set of all vertices of γ be $V(\gamma)$ and the set of all oriented edges of γ be $E(\gamma)$. Since we have an integer lattice on Σ, we have $V(\gamma) \equiv \mathbb{Z}^3$.

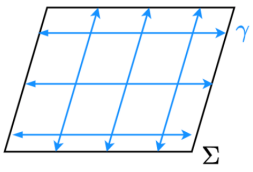

Figure 8: The integer lattice

Consider a single vertex v. (In our lattice, all vertices are 6-valent). The vertices and edges are labelled as shown in

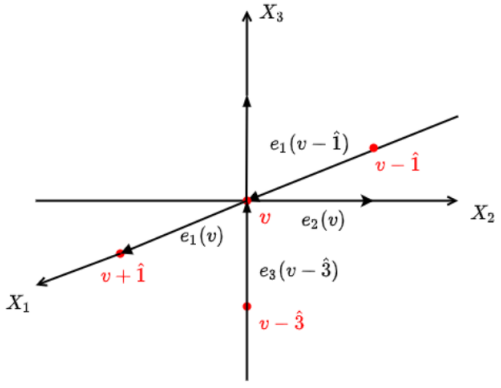

Figure 9: Vertex labelling

the figure 9. In the figure the edges have been labelled in black and the vertices have been labelled in red. The edges

are labelled as $e_i(v')$ where i is the direction of its orientation and v' is the vertex it originates from. Note that, the directions are labelled 1, 2 and 3 instead of i, j and k.

Now, we have to discretise our smooth fields $A_a^i(x)$ and $\widetilde{E}_i^a(x)$. Firstly, for each edge e, we define a holonomy, $h(e) \in SU(2)$, an element of the group SU(2). This is the same as the Wilson line definition. Secondly, we define $p^j(e) \in \mathbb{R}^3$. This is a vector, called the gauge covariant flux.

$$\begin{array}{ccc} \text{Smooth Fields} & & \text{Lattice Variables} \\ A_a^i(x) & \longrightarrow & h(e) \in SU(2) \\ \widetilde{E}_i^a(x) & \longrightarrow & p^j(e) \in \mathbb{R}^3 \end{array} \Bigg\} \forall\, e \in E(\gamma)$$

The holonomy and flux are the lattice variables replacing the connection variables $A_a^i(x)$ - the gauge connection, and $\widetilde{E}_a^j(x)$ - the densitised triad.

Lattice discretisation approximates a field theory by a system with finite number of degrees of freedom, and we should be able to recover the field theory by lattice refinement (that is, what we call the continuum limit). We should be able to obtain the smooth fields from the discretised variables if we infinitely refine the lattice.

Note that everything is still classical. We have not performed any quantisation yet. We define classical lattice variables first and then quantise them.

12.1 Holonomy

Given the Ashtekar connection 1-form $A_a^i(x)$, (Recall that $A_a^i(x)$ is an SU(2) gauge potential) and an oriented edge e, that is, the edge e is just a curve, a map from the interval $[0,1] \longrightarrow \Sigma$ such that $s \longrightarrow e(s) \in \Sigma\ \forall\ s \in [0,1]$. We consider this map to be simple/analytic and a tangent vector of e is denoted by

$$\left(\frac{d}{ds}\right)^a \equiv \dot{e}^a = \sum_{\mu=1}^{3} \frac{dx^\mu}{ds} \left(\frac{d}{dx^\mu}\right)^a \tag{12.1}$$

$$\text{where } \dot{e}^a = \frac{d}{ds}[e(s)]$$

Now, we define the holonomy along the edge e $h(e)$ as,

$$h(e) := P\ exp\left[\int_0^1 ds A_a^i \dot{e}^a \frac{\tau_i}{2}\right] \text{ where } \tau_i = -i\sigma_i \tag{12.2}$$

where σ_i are the Pauli matrices. We have a path ordering operator in the front because the exponent has matrices, and matrices are not commutative. This path ordering is the reason why we need an orientation for the edge to be able to define the holonomy, because in order to define the holonomy, we need to know where $e(o)$ is and where $e(1)$ is.

$$h(e) := P\ exp\left[\int_0^1 ds A_\mu^i \frac{dx^\mu}{ds} \frac{\tau_i}{2}\right] \tag{12.3}$$

where the matrix $A_a = A_a^i \dfrac{\tau_i}{2} \in \mathfrak{su}(2)$ lie algebra because the τ_i's (the Pauli matrices) are just the generators of the $\mathfrak{su}(2)$ lie algebra.

Therefore, A_a is a lie-algebra valued 1-form (This is the basis of gauge theory. The connection variable or the gauge potential is always a lie algebra valued 1-form). And the holonomy is a path ordered integral of this 1-form in the exponential. Therefore $h(e) \in SU(2)$.

Theorem: The holonomy h(e) satisfies the following properties:

(1) Given edges e_1 and e_2, we can connect these two edges. After we compose these two edges. After we compose

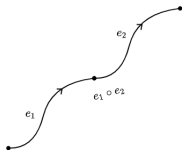

Figure 10: Composition of Holonomies

these two edges we obtain a new edge that we will call $e_1 \circ e_2$.

$$h(e_1 \circ e_2) = h(e_1) \cdot h(e_2) \tag{12.4}$$

where the dot denotes the group operation of $SU(2)$.

(2) If you reverse the orientation of e, and call it e^{-1},

$$h(e^{-1}) = h^{-1}(e) \tag{12.5}$$

where $h^{-1}(e)$ denotes the inverse of the group element $h(e)$.

Properties of (1) and (2) imply that h is a homomorphism from space of curves or edges to the gauge group $SU(2)$.

(3) h(e) is a solution of the differential equation,

$$\frac{d}{ds}h(s) = h(s)A(s) \tag{12.6}$$

where $A(s) = A_a^i(s)\dot{e}^a \frac{\tau_i}{2}$ Equation (12.6) is the parallel transport equation. Now, this is an ordinary differential equation and to solve it we need to set an initial condition, given by,

$$h(s=0) = \mathbb{1}_2 \tag{12.7}$$

Thus, since this differential equation has an initial condition, we can uniquely determine its solution. From this solution $h(s)$, we define our holonomy $h(e)$ as

$$h(e) = h(s=1) = h(1) \tag{12.8}$$

(4) Gauge Transformation: A holonomy is a function of the gauge potential. The gauge potential is not gauge invariant, it changes under a gauge transformation. Thus the holonomy changes too. Recall that we have $A_a(x) = A_a^i(x)\frac{\tau_i}{2}$. Under a gauge transformation,

$$A_a(x) \longrightarrow g(x)A_a(x)g^{-1}(x) - [\partial_a g(x)]g^{-1}(x) \tag{12.9}$$

where $g(x)$ is a group valued function on the spatial slice Σ. By plugging in this definition of gauge transformation into the definition of holonomy, we get,

$$h(e) \longrightarrow g(s(e)) \cdot h(e) \cdot g^{-1}(t(e)) \tag{12.10}$$

where $s(e) = e(s = 0)$ and is called the source of edge e and $t(e) = e(s = 1)$ is called the target of edge e. $g(x)$ is a smooth function but since the holonomy is a discrete variable, its gauge transformation depends only on the value of the function at the two points, the source and the target.

□□□

If, for some $v \in \sigma, s(e) = t(e) = v$, that is, we have a closed loop (which we do not have in our lattice), then

$$h(e) \longrightarrow g(v) \cdot h(e) \cdot g^{-1}(v) \tag{12.11}$$

Then if we define a variable trace of $h(e)$ and under a gauge transformation, $Tr[h(e)]$ remains invariant. This is one of the main reasons why we use holonomy in gauge theory. The $Tr[h(e)]$ is called a Wilson loop variable. And $h(e)$ is called the Wilson line variable.

12.2 Gauge Covariant Flux

Consider the spatial slice Σ and the lattice γ defined on it. Once we have the lattice γ, we can build the dual lattice γ^*, as shown in the figure below. In the figure, γ is our cubic lattice and γ^* is our dual lattice. The lattice has been shown in black and the dual lattice has been shown in red in the figure.

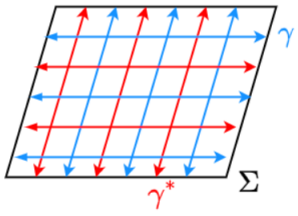

Figure 11: The dual lattice γ^*

Every vertex is dual to a cube (in our 3-dim spatial slice, 0-dim is dual to 3-dim) and every edge is dual to a face of the cube (1-dim is dual to 2-dim). The gauge covariant flux is defined on the surface dual to the edge.

Let us call the point of intersection of the edge with the dual surface as p. This dual surface is called S_e (the surface dual to the edge e).

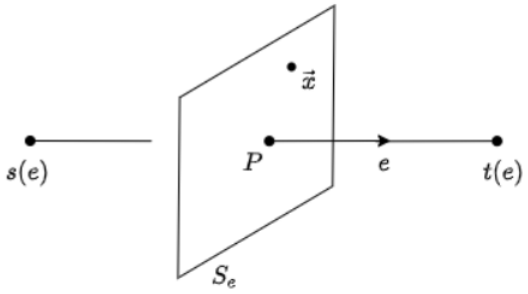

Figure 12: The edge e, the dual surface S_e and the point of intersection p

Now, given an edge e and its dual surface S_e, we define the gauge covariant flux. Thus flux is a function of both the connection variable and the densitised triad. ($P^j(e)$ is a function of both A^i_a and E^a_i).

Given e and S_e, $\forall\ \vec{x} \in S_e$, $\exists$ a path connecting $p = S_e \cap e$ to $\vec{x}$ (that is given e and S_e, we pick an arbitrary point on S_e and take the path connecting that p to that p).

Now we have a path $s(e)$ to $\vec{x}$ (that is $s(e)$ to p along the edge and p to $\vec{x}$ in the surface S_e). We call this complete path from $s(e)$ to $\vec{x}$ as $\rho_e(\vec{x})$, (as ρ depends on both e and $\vec{x}$) shown in red in the figure below. Once we have $\rho_e(\vec{x})$, we can

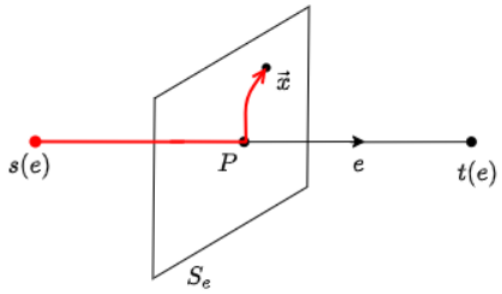

Figure 13: The path $\rho_e(\vec{x})$

define the holonomy along $\rho_e(\vec{x})$, given by $h(\rho_e(\vec{x}))$. Now we define our gauge covariant flux,

$$P^j(e) := -\frac{1}{2\beta} Tr\left[\tau^j \oint_{S_e} \varepsilon_{abc}\, dx^a \wedge dx^b h(\rho_e(\vec{x})) E^c_k(\vec{x}) \tau^k h^{-1}(\rho_e(\vec{x}))\right] \tag{12.12}$$

where $dx^a \wedge dx^b$ is the integrating 2-form, $E^c_k(\vec{x})\tau^k$ represents the two triads as a lie algebra matrix in $SU(2)$, and the $h(\rho_e(\vec{x}))E^c_k(\vec{x})\tau^k h^{-1}(\rho_e(\vec{x}))$ is the adjoint representation, the adjoint action of the $SU(2)$ group on the $\mathfrak{su}(2)$ lie algebra. Alternatively, we can just think of $E_k\tau^k$ as a matrix with h and h^{-1} as just matrix multiplication. So this is just a matrix valued function that we integrate over the surface S_e. We use an integrating 2-form to integrate over the surface.

If we take $({}^*E)$ to be the 2-form dual to the 1-form E_a, we have

$$({}^*E)_{ab} = \varepsilon_{abc}E^c \text{ where } E^c = E^c_k\tau^k \tag{12.13}$$

Then, the gauge covariant flux can be defined in terms of this dual as

$$P^j(e) := -\frac{1}{2\beta} Tr\left[\tau^j \oint_{S_e} h(\rho_e(\vec{x}))\ [^*E(\vec{x})]\ h^{-1}(\rho_e(\vec{x}))\right] \tag{12.14}$$

This integral, and hence the flux is coordinate independent.

Theorem :

(1) P^j is gauge covariant.

$$P(e) \equiv P^j(e)\frac{\tau^j}{2} \in \mathfrak{su}(2) \text{ lie algebra} \tag{12.15}$$

Since P^j is a function of both $A^i_a(x)$ and $\widetilde{E}^a_i(x)$ a gauge transformation of P^j must take into account gauge transformations of both $A^i_a(x)$ and $\widetilde{E}^a_i(x)$. We have

$$E^b_k \tau^k(x) = g(\vec{x}) E^b_k \tau^k(x) g^{-1}(\vec{x}) \tag{12.16}$$

$$h(\rho_e(\vec{x})) \longrightarrow g(s(e)) h(\rho_e(\vec{x})) g^{-1}(\vec{x}) \tag{12.17}$$

From equations (12.9) and (12.16), we can see that both $A^i_a(x)$ and $\widetilde{E}^a_i(x)$ transform in the adjoint representation ($A^i_a(x)$ transforms like a connection). Hence, P^j also transforms in the adjoint representation. The gauge transformation of P^j is given by,

$$P^j(e) = g(s(e)) P^j(e) g^{-1}(s(e)) \tag{12.18}$$

The gauge transformation of P^j depends only on the value of g at $s(e)$ and not at $t(e)$ because $\rho_e(\vec{x})$ depends only on $s(e)$ and not at $t(e)$. From equation (12.18) we say that $P^j(e)$ transforms as a vector in the adjoint representation of $SU(2)$.

(2) If the coordinate length of e and coordinate area of S_e are μ and μ^2, then in the continuum limit (that is if you infinitely refine the lattice),

$$\lim_{\mu\to 0} p^j(e) = \frac{2\mu^2}{\beta} E^a_j(s(e)) + O(\mu^3) \tag{12.19}$$

(3)

$$P^j(e^{-1}) = \frac{1}{2} Tr\left[\tau^j h(e) P^i(e) \tau^i h^{-1}(e)\right] \tag{12.20}$$

□□□

Now, in the beginning, we had smooth fields ($A^i_a(x)$, $\widetilde{E}^a_i(x)$), (that is, the Ashtekar connection and the densitised triad), which have smooth SU(2) gauge transformations. On the lattice, after discretisation, we have the lattice fields $(h(e), P^j(e))$. These variables form the space,

$$(h(e_i), P^j(e_i)) \in (SU(2) \times \mathbb{R}^3)^{|E(\gamma)|} \qquad \forall \quad i = 1,2,3...|E(\gamma)| \tag{12.21}$$

where, $|E(\gamma)|$ = number of edges in the graph γ, because $h(e) \in SU(2)$ and $P^j \in \mathbb{R}^3$. $SU(2) \times \mathbb{R}^3$ is usually written as a cotangent bundle of SU(2).

$$SU(2) \times \mathbb{R}^3 \cong T^*(SU(2)) \tag{12.22}$$

Lattice variables have lattice gauge transformations which are the gauge transformations obtained by discretising the smooth $SU(2)$ gauge transformations. The gauge transformations we use now is $g : V(\gamma) \longrightarrow SU(2)$, that is, $g(v) \in SU(2)\ \forall\ v \in V(\gamma)$. Thus, the set of all gauge transformations is $[SU(2)]^{|V(\gamma)|}$.

A key point that must be emphasised here is that this discretisation is background independent. This is because all of our discretisations do not depend on any metric. It is coordinate independent and our discretisation does not assume the existence of a metric. It has to be like that for gravity because the gravitational field is background independent. Hence our quantisation procedure must also be completely background independent, since we are trying to obtain a background independent theory of quantum gravity.

12.3 Holonomy-Flux Algebra

The Poisson Algebra between the lattice variables

$$\{h(e), h(e')\} = 0 \tag{12.23}$$

$$\{P^j(e), h(e')\} = \kappa\delta_{ee'}\frac{\tau^j}{2}h(e) \tag{12.24}$$

where κ is the gravitational constant and the $\delta_{ee'}$ makes sure that the Poisson bracket is non-zero if and only if the dual surface over which $P^j(e)$ integrates, intersects with e. Since $h(e)$ is a matrix, if we include the matrix indices A,B,C,D= 1,2, we get

$$\{h_{AB}(e), h_{CD}(e')\} = 0 \tag{12.25}$$

$$\{P^j(e), h_{AB}(e')\} = \kappa\delta_{ee'}\left(\frac{\tau^j}{2}h(e)\right)_{AB} \tag{12.26}$$

And, finally, we have,

$$\{P^j(e), P^k(e')\} = -\kappa\delta_{ee'}\varepsilon^{jkl}P^l(e) \tag{12.27}$$

These Poisson brackets are derived from the Poisson brackets of $A^i_a(x)$ and $\widetilde{E}^a_i(x)$ smooth fields. You can see that $h(e)$ and $P^j(e)$ are variables conjugate to one another. We can also see this from the continuum limit.

Another interesting point of the Holonomy-Flux algebra is that equation (12.27) is like an analogue of the angular momentum relation

$$[J^i, J^j] = i\hbar\varepsilon^{ijk}J^k \tag{12.28}$$

Thus you can see that the quantisation of the gauge covariant flux is just angular momentum operators.

Now the phase space on our lattice is P_γ, where P_γ is given by,

$$p_\gamma \cong [T^*(SU(2))]^{|E(\gamma)|}/[SU(2)]^{|V(\gamma)|} \tag{12.29}$$

where $T^*(SU(2))$ is the cotangent bundle of $SU(2)$. This is our total space modulo the space of the gauge transformations. The simplectic structure of the space is given by the Holonomy Flux algebra (the Poisson bracket structure of the holonomy and flux variables).

12.4 Quantisation

In our Holonomy-Flux phase space our wave functions are functions of all holonomies on the lattice γ, (that is, the lattice γ has many edges, and your wave function must depend on the holonomies along all the edges.) Thus, the wave function must be of the form,

$$f(h(e_1),....,h(e_{|E(\gamma)|})) \tag{12.30}$$

Now our Hilbert space is the space of these functions, which we call $\mathscr{H}_\gamma^0$. It is defined as

$$\mathscr{H}_\gamma^0 = L^2(SU(2),d\mu_H)^{|E(\gamma)|} \tag{12.31}$$

where $d\mu_H$ is that Haar measure on the SU(2) group. For every edge you have one holonomy and one Hilbert space for every holonomy. Here the power $|E(\gamma)|$ denotes a product, a tensor product of each of those Hilbert spaces, $|E(\gamma)|$ times. Since you have an SU(2) measure on each edge the entire Hilbert space is raised to $|E(\gamma)|$.

As a convention, we take holonomy to be our position variable and flux to be our momentum variable and not the other way around. This is because the flux variables do not commute,

$$[\hat{P}^j(e),\hat{P}^k(e')] = -il_p^2\delta_{ee'}\varepsilon^{jkl}\hat{P}^l(e) \tag{12.32}$$

But holonomies are commutative. The wavefunction, would then be defined on a commutative space. But, if we want to define our wave function as a function of fluxes, then this function would be defined on a non commutative space. Hence, by convention, we choose holonomy to be our position variable and flux to be our momentum variable. But quantisation can be done either way and either way we choose to define our variables the formalism will be completely equivalent by the Stone-von Neumann uniqueness theorem.

12.5 Spin Network Basis

Now, lets consider just the Hilbert space on one edge, given by $L^2(SU(2),d\mu_H)$. We need a basis for this Hilbert space. Once we understand this Hilbert space, we will be able to understand $L^2(SU(2),d\mu_H)^{|E(\gamma)|}$, the Hilbert space on the entire lattice.

An orthogonal basis for $L^2(SU(2),d\mu_H)$ are the Wigner D-functions (matrix elements of the Wigner D-matrices), given by

$$D^j_{mn}(h) \text{ where } j = 0,\frac{1}{2},1,\frac{3}{2},... \tag{12.33}$$

m and n are magnetic quantum numbers. $m,n = -j,-j+1,...,j-1,j$.

The $D^j_{mn}(h)$ has two indices m and n, it is a matrix in the spin j space. $|j,m\rangle$ are the basis vectors of the spin j Hilbert space called V_j, which are the spin j irreducible representations.

We have the orthogonality relation given by,

$$\int d\mu_H(h)\,\overline{D^j_{mn}(h)}D^{j'}_{m'n'}(h) = \frac{1}{2j+1}\delta_{jj'}\delta_{mm'}\delta_{nn'} \tag{12.34}$$

Therefore, $\{D^j_{mn}(h) \mid j \in \frac{\mathbb{Z}_+}{2} \cup \{0\}\ ,\ m,n = -j,-j+1,...j\}$ is an orthogonal basis for $L^2(SU(2),d\mu_H)$. Therefore,

$$\forall\ f \in L^2(SU(2),d\mu_H), \quad f(h) = \sum_{j,m,n} f_{j,m,n} D^j_{mn}(h) \tag{12.35}$$

where the $f_{j,m,n}$ are the complex coefficients, that is, $f_{j,m,n} \in \mathbb{C}$. Now, the orthogonal basis in total Hilbert space $\mathscr{H}^0_\gamma = L^2(SU(2),d\mu_H)^{|E(\gamma)|}$ is

$$\prod_{e\in E(\gamma)} D^{j_e}_{m_e n_e}(h_e) \equiv T_{\vec{j},\vec{m},\vec{n}}(\vec{h}) \tag{12.36}$$

where the vector $\vec{j}$ is the sequence of j's on all edges.

The inner product of $\mathscr{H}^0_\gamma$ is given by

$$\int \prod_{e\in E(\gamma)} d\mu_H(h_e)\ \overline{T_{\vec{j},\vec{m},\vec{n}}(\vec{h})} T_{\vec{j}',\vec{m}',\vec{n}'}(\vec{h}') = \prod_{e\in E(\gamma)} \frac{1}{2j_e+1} \delta_{j_e j'_e} \delta_{m_e m'_e} \delta_{n_e n'_e} \tag{12.37}$$

$$\forall\ f \in \mathscr{H}^0_\gamma, \quad f(\vec{h}) = \sum_{\vec{j},\vec{m},\vec{n}} f_{\vec{j},\vec{m},\vec{n}} T_{\vec{j},\vec{m},\vec{n}}(\vec{h}) \tag{12.38}$$

Now, these functions $f(\vec{h}) \in \mathscr{H}^0_\gamma$ are not gauge invariant, because

$$f(\vec{h}) = f(h(e_1),...h(e_{E(\gamma)})) \in \mathscr{H}^0_\gamma \tag{12.39}$$

f is a function of the holonomies on each edge and the holonomies themselves are not gauge invariant. This zero denotes a non-gauge invariant Hilbert space, or a Hilbert space of non-gauge invariant wave functions. Now we have to define a Hilbert space of gauge invariant functions.

To obtain the gauge transformation of a function $f(\vec{h}) \in \mathscr{H}^0_\gamma$ we simply plug in the gauge transformation of a holonomy. Now

$$f(\{h_e | e \in E(\gamma)\}) \longrightarrow f(\{g(s(e))\ h_e\ g^{-1}(t(e)) | e \in E(\gamma)\}) \tag{12.40}$$

this is the gauge transformation for wave functions. But as we can see the only values of the gauge field that we use in the transformation are the guage fields at the vertices. So in LQG on a lattice, the gauge transformation uses values of the gauge field only at the vertices.

Gauge transformations lead to physically equivalent states. So we look for gauge invariant functions and make a new Hilbert space $\mathscr{H}_\gamma$ that is a subspace of the old Hilbert space $\mathscr{H}^0_\gamma$, that is, $\mathscr{H}_\gamma \subset \mathscr{H}^0_\gamma$. This $\mathscr{H}_\gamma$ that we obtain gives the quantisation of our phase space P_γ given in equation (12.29).

We are going to construct gauge invariant functions by taking linear combinations of the basis vectors, $T_{\vec{j},\vec{m},\vec{n}}(\vec{h})$. Let's consider a vertex (6-valent in our case) and for simplicity let's consider all the edges $e_1,...e_6$ to be outgoing. Now the graph has other edges and vertices only on the neighbourhood of a single vertex

$$T_{\vec{j},\vec{m},\vec{n}}(\vec{h}) = \prod_{e=1}^{6} D^{j_e}_{m_e n_e}(h_e).... \tag{12.41}$$

where the dots in the end denote the other edges and vertices of the graph.

Under a gauge transformation,

$$\prod_{e=1}^{6} D^{j_e}_{m_e n_e}(h_e)....\ \longrightarrow \prod_{e=1}^{6} D^{j_e}_{m_e n_e}(g_v h_e g^{-1}_{t(e)})....\tag{12.42}$$

$$T_{\vec{j},\vec{m},\vec{n}}(\vec{h}) \longrightarrow \prod_{e=1}^{6} D^{j_e}_{m_e k_e}(g_v) D^{j_e}_{k_e l_e}(h_e) D^{j_e}_{l_e n_e}(g^{-1}_{t(e)})\tag{12.43}$$

Thus we see that the basis is not gauge invariant. Now we consider linear combinations of $T_{\vec{j},\vec{m},\vec{n}}(\vec{h})$. The first part of the term the $D^{j_e}_{m_e n_e}(g_v)$ is common to all six edges. The j_e, m_e and k_e depend on e but g_v does not since it is a gauge transformation. We will sum over the m_e index. Consider a tensor $C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6}$.

$$C_{j_1 \ldots j_6} \in V_{j_1} \otimes \ldots \otimes V_{j_6}\tag{12.44}$$

$C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6}$ belongs to the tensor product of vector spaces (the m_i indices are the vector indices for each vector belonging to V_{j_i}). Basically,

$$C_{j_1 \ldots j_6} = \sum_{\vec{m}} C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6} |j_1, m_1\rangle \otimes \ldots |j_6, m_6\rangle\tag{12.45}$$

So, this is a tensor, $C_{j_1 \ldots j_6}$ is a tensor which couples 6 angular momenta.

Up until now, $C_{j_1 \ldots j_6}$ is an arbitrary tensor. Now, let's take a linear combination of the tensor with respect to the basis. Consider the linear combinations.

$$\begin{aligned}\sum_{\vec{m}} C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6} \prod_{e=1}^{6} D^{j_e}_{m_e n_e}(h_e) \qquad &\longrightarrow \\ \sum_{\vec{m}} C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6} \prod_{e=1}^{6} D^{j_e}_{m_e k_e}(g_v) D^{j_e}_{k_e l_e}(h_e) D^{j_e}_{l_e n_e}(g^{-1}_{t(e)})&\end{aligned}\tag{12.46}$$

Now, we can make this linear combination gauge invariant at the vertex v if,

$$\sum_{\vec{m}} C_{j_1 \ldots j_6}^{\ m_1 \ldots m_6} \prod_{e=1}^{6} D^{j_e}_{m_e k_e}(g_v) = C_{j_1 \ldots j_6}^{\ k_1 \ldots k_6}\tag{12.47}$$

If (12.47) holds locally at vertex v, then the linear combination is gauge invariant at v. If such an expression holds at every vertex then you have gauge invariants at every vertex and since all you gauge transformations are at vertices, the entire wave function/state will be gauge invariant if (12.47) holds at every vertex. (12.47) is the definition of an invariant tensor, that is $C_{j_1 \ldots j_6}$ is an invariant tensor (invariant under the action of SU(2)) in the tensor product space $V_{j_1} \otimes \ldots \otimes V_{j_6}$.

$D^j_{mn}(g)$ is a representation of $SU(2)$, that is, $\forall g \in SU(2)$, $D^j(g)$ gives a map from V_j to itself. Thus, $\forall g \in SU(2)$, $D^{j_1}(g) \otimes \ldots \otimes D^{j_6}(g)$ gives a map from $V_{j_1} \otimes \ldots \otimes V_{j_6}$ to itself. This is a representation of SU(2) in $V_{j_1} \otimes \ldots \otimes V_{j_6}$. $\prod_{e=1}^{6} D^{j_e}_{m_e k_e}(g_v)$ is a tensor product. This acts on a vector $C_{j_1 \ldots j_6} \in V_{j_1} \otimes \ldots \otimes V_{j_6}$ to give another vector in this space. A tensor (an object that lies in the tensor product space $V_{j_1} \otimes \ldots \otimes V_{j_6}$) that is invariant under this SU(2) action is called an invariant tensor.

Lemma: On each representation V_{j_e}, $\exists$ an angular momentum operator $\vec{J}_e$ (because this is an $SU(2)$ representation space, we can define an angular momentum operator)

i) $C_{j_1\dots j_6}$ is an invariant tensor $\Leftrightarrow$ Zero total angular momentum state.

$$C_{j_1\dots j_6} \text{ is invariant } \Leftrightarrow \sum_{e=1}^{6} \vec{J}_e \, |C_{j_1\dots j_6}\rangle = 0 \tag{12.48}$$

where we write

$$|C_{j_1\dots j_6}\rangle = C_{j_1\dots j_6}^{\;\;m_1\dots m_6} \, |j_1, m_1\rangle \otimes \dots |j_6, m_6\rangle \tag{12.49}$$

Thus, $|C_{j_1\dots j_6}\rangle$ is a state that belongs to the vector space, $V_{j_1} \otimes \dots \otimes V_{j_6}$. A state $|C_{j_1\dots j_6}\rangle$ satisfying (12.47), or equivalently (12.48) is called an intertwiner. These states are also called invariant states or zero angular momentum states. Such states span the invariant subspace $Inv(V_{j_1} \otimes \dots \otimes V_{j_N})$.

Now, equation (12.48) is like an eigenvalue equation, all states with the total angular momentum eigenvalue as zero, is a subspace of the total space.

$$Inv(V_{j_1} \otimes \dots \otimes V_{j_N}) \subset V_{j_1} \otimes \dots \otimes V_{j_N} \tag{12.50}$$

where $Inv(V_{j_1} \otimes \dots \otimes V_{j_N})$ is the space of interwiners.

So, now we see that in order to construct a gauge invariant wave function, we just need to insert this kind of interwiners, at all the vertices and contract them with the basis, that is, we make a linear combination of the sort,

$$\sum_{\vec{m}} C_{j_1\dots j_6}^{\;\;m_1\dots m_6} \prod_{e=1}^{6} D^{j_e}_{m_e k_e}(h_e) \quad \text{where} \quad C_{j_1\dots j_6} \in Inv(V_{j_1} \otimes \dots \otimes V_{j_6}) \tag{12.51}$$

Now we construct a basis of gauge invariant states, given by,

$$T_{\gamma,\vec{j},\vec{i}}(\vec{h}) = \prod_{e\in E(\gamma)} D^{j_e}(h_e) \prod_{v\in V(\gamma)} C_{\vec{j},i_v} \tag{12.52}$$

where $C_{\vec{j}} \in V_{j_1} \otimes \dots \otimes V_{j_N}$. The m and n indices contract with the intertwiners of the respective vertices as ashown in the figure. The dimensions of the $\vec{i} = |V(\gamma)|$ and the $\vec{j} = |E(\gamma)|$.

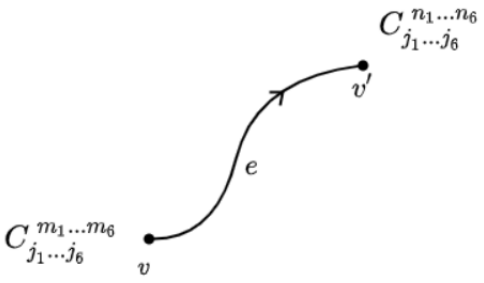

Figure 14: The m and n indices contract with intertwiners of vertices v and v'

The i_v is the interwiner basis label. We can define an inner product in the space $Inv(V_{j_1} \otimes \dots \otimes V_{j_N})$ as,

$$\langle C_{\vec{j}} i_v | C_{\vec{j}} i'_v \rangle = \delta_{i_v i'_v} \tag{12.53}$$

The Hilbert space $Inv(V_{j_1} \otimes \dots \otimes V_{j_N})$ is always finite dimensional.

We have assumed that our lattice is a closed graph, that is, it has no dangling edges (because graphs with dangling edges cannot have gauge invariance because an intertwiner cannot be defined on the dangling end of the edge). This is because the equation (12.48) is satisfied only by the $|0\rangle$ state at the dangling end, it does not have a non trivial solution at the dangling end. Hence, if you have a dangling edge your state cannot be gauge invariant. Hence in order to have a gauge invariant wave function, you need to have a closed graph.

Now, $\{T_{\gamma,\vec{j},\vec{i}}\}$ is the orthogonal basis of the Hilbert space of gauge invariant wave function given by $\mathscr{H}_\gamma$. $\{T_{\gamma,\vec{j},\vec{i}}\}$ is also called the spin network basis. Inner product of these basis states in $\mathscr{H}_\gamma^0$ or $\mathscr{H}_\gamma$ (the inner product exists both in $\mathscr{H}_\gamma^0$ and $\mathscr{H}_\gamma^0$) is given by,

$$\langle T_{\gamma,\vec{j},\vec{i}}|T_{\gamma,\vec{j}',\vec{i}'}\rangle = \prod_{e\in E(\gamma)} \frac{1}{2j_e+1}\delta_{\vec{j}\vec{j}'}\delta_{\vec{i}\vec{i}'} \tag{12.54}$$

Inner product in $\mathscr{H}_\gamma^0$ is given by,

$$\langle f|f'\rangle = \int \prod_e d\mu_H(h_e)\overline{f(h_e)}f'(h_e) \tag{12.55}$$

The spin network basis is labelled by $(\gamma,\vec{j},\vec{i})$ is an oriented graph (lattice in our case), that is, it has oriented edges. The edges have spins j_e and the vertices have interwiners i_v.

Now we have a Hilbert space H_γ with a basis, the spin network basis. Now, we need operators to act on these states.

12.6 Holonomy and Flux Operators

The operators are $\hat{h}(e)$ and $\hat{P}^j(e)$ These operators act on $\mathscr{H}_\gamma^0$ and not on $\mathscr{H}_\gamma$, since the operators themselves are not gauge invariant. But we use these operators to construct gauge invariant operators. Only gauge invariant operators can denote physical observables.

The holonomy operator is like the position operator, it is multiplied. Since holonomy is an element of SU(2) it is a 2×2 matrix. Its action on a function $f(\vec{h})$ is given by,

$$\hat{h}_{AB}(e)f(\vec{h}) := h_{AB}(e)f(\vec{h}) \tag{12.56}$$

where $f(\vec{h})$ is a function of holonomy. The action of the holonomy operator on the function is just its multiplication by h.

Now, the action of the flux operator on the function $f(\vec{h})$ is defined as,

$$\hat{P}^j(e)f(\vec{h}) := i\frac{l_p^2}{2}\hat{R}_e^j f(\vec{h}) \tag{12.57}$$

where $l_p^2 = \hbar\kappa$ and $\hat{R}_e^j$ is the right invariant vector field on SU(2). It acts only on the e^{th} component of the vector (since $\hat{P}^j(e)$ is an operator defined on the edge e, hence it does not act on the other components of the vector). Recall that a vector field is a map from function to function, that is, it takes a function to another function. Thus the vector field is like an operator.

The commutation relations between these operators can be obtained either by directly using their definitions or by quantising their Poisson algebra.

$$[\hat{h}_{AB}(e), \hat{h}_{CD}(e')] = 0 \tag{12.58}$$

$$[\hat{P}^j(e), \hat{h}_{AB}(e')] = il_p^2 \delta_{ee'} \left(\frac{\tau^j}{2} \hat{h}(e) \right)_{AB} \tag{12.59}$$

$$[\hat{P}^j(e), \hat{P}^k(e')] = -il_p^2 \delta_{ee'} \varepsilon^{jkl} \hat{P}^l(e) \tag{12.60}$$

Note that at $e = e'$, the third commutation relation becomes an analog of the angular momentum commutation relation in quantum mechanics. Thus the flux operator is like an angular momentum operator and the wave functions are like spin network states.

$$\hat{P}^j \sim \hat{J}^j \quad ; \quad f(\vec{h}) \sim \text{ spin network states} \tag{12.61}$$

The origin of the spins is the third commutation relation and its analog in quantum mechanics. This analog is because of the commutation relation of the right invariant vector field. If $e = e'$,

$$[\hat{R}_e^j, \hat{R}_e^k] = 2\varepsilon^{jkl} \hat{R}_e^l \tag{12.62}$$

Thus, we have the correspondence,

$$-i\frac{\hat{R}^j}{2} \equiv \hat{J}^j \tag{12.63}$$

This happens because the right invariant vector field is also a certain kind of representation. As you can see from the formula, the right invariant vector field is generated by τ^j. Thus, the commutation relations of τ translate to the commutation relations of $\hat{R}_e^j$.

12.7 Area and Volume Operators

We are going to use the holonomy and flux operators to construct gauge invariant operators – the area operator and the volume operator. These operators are important because they quantise the geometry of space time. Since, classically gravitational field is just the geometry of a curved space time, quantum mechanically, we would want the operators of the gravitational field to be operators of geometry.

Recall that our manifold $\mathscr{M} = \Sigma \times \mathbb{R}$. The operators we discuss here will be quantising space and not spacetime. So here we are talking about the geometry of space. This is because, all the fields that we've described so far are fields on space,like holonomy and flux. Our holonomy and flux variables are discretisations of fields on the spatial slice Σ. Since we are quantising only space and space is Riemannian manifold, the area and the volume operators make up what we sometimes called Quantum Riemannian Geometry.

Area Operator

We first discretise the classical expression for area and then quantise this discrete variable.

Consider a 2-dim spatial slice, S, in our 3-dim space Σ. S is a 2-dim surface. The surface S whose area we are trying to measure is shown in black, the lattice is shown in blue and the dual lattice that we use to approximate the surface

S is shown in red, in the figure below. (Since we have suppressed 1-dim, the 2-dim slice S looks like a curve in our

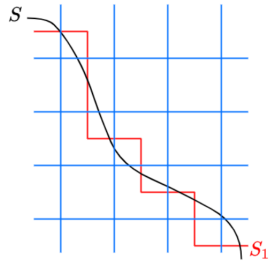

Figure 15: Approximation of a surface S

figure).

Recall that the useful quantity for the lattice is the flux. The flux is the smear of the densitised triad $\widetilde{E}^a_i$ on the dual surface. We use this dual surface to discretise our surface S. This approximation has been done in red in the figure. The approximation is similar to approximating a function using a step function. Lets call this surface as S_1. S_1 is not exactly S, but, it is an approximation. If we infinitely refine the lattice, S_1 will converge to S.

Now, S can be approximated as the sum of S_{e_i}'s as,

$$S \simeq \sum_{i=1}^{n} S_{e_i} \quad \text{such that} \quad e_i \cap S \neq \phi \tag{12.64}$$

where where we sum over the S_{e_i}'s such that $e_i \cap S \neq \phi$. S can be discretised by using the faces dual to the edges that intersect with it.

Classically, the area of a smooth surface S is given by

$$Ar(S) = \int_S d^2\sigma \ \sqrt{detq} \tag{12.65}$$

where σ is a coordinate on S and q is the induced metric on S. On rewriting the expression for area in terms of the continuum variables (the connection and the densitised triad), we obtain,

$$Ar(S) = \int_S d^2\sigma \ \sqrt{n_a E^a_j(\sigma)\ n_b E^b_j(\sigma)} \tag{12.66}$$

where n_a is the co-normal of the 2-dim surface S. Since our spatial slice Σ is 3-dim the 2-dim surface S is a hypersurface that has a unique co-normal.

Now, say, n edges from our graph intersect our surface S. From equation (12.64), we get

$$Ar(S) \ \simeq \ \sum_{i=1}^{n} Ar(S_{e_i}) \tag{12.67}$$

where the sum is taken over the surfaces dual to the edges that intersect with S.

$$Ar(S) \ \simeq \ \sum_{i=1}^{n} \int_{S_{e_i}} d^2\sigma \ \sqrt{n_a E^a_j(\sigma)\ n_b E^b_j(\sigma)} \tag{12.68}$$

We assume the lattice to be very refined, we are close to the continuum limit. Under this assumption, we make another approximation,

$$\int_{S_{e_i}} d^2\sigma \, \sqrt{n_a E^a_j(\sigma) \, n_b E^b_j(\sigma)} \simeq \sqrt{\int_{S_{e_i}} d^2\sigma \, n_a E^a_j(\sigma) \int_{S_{e_i}} d^2\sigma \, n_b E^b_j(\sigma)} \tag{12.69}$$

Say we put coordinates such that σ^1 and σ^2 lie on the surface S and σ^3 is the co-normal. In this case, $n_a = \sigma^3$. Now the surface is the set of all point such that $\sigma^3 = 0$. Now, lets consider,

$$\begin{aligned} \int_{S_{e_i}} d^2\sigma n_a E^a_j(\sigma) &= \int_{S_{e_i}} d^2\sigma \, E^3_j(\sigma) \\ &= \frac{1}{2} \int_{S_{e_i}} dx^a \wedge dx^b \varepsilon_{abc} E^c_j \end{aligned} \tag{12.70}$$

We move from the σ coordinates which are special coordinates attached to the surface S to arbitrary coordinates x^a. Then, we write,

$$\int_{S_{e_i}} d^2\sigma n_a E^a_j(\sigma) = -\frac{1}{4} Tr \left[\tau^j \int_{S_{e_i}} dx^a \wedge dx^b \varepsilon_{abc} E^c_k \tau^k \right] \tag{12.71}$$

If you take the trace of $\tau^j \tau^k$, then you can obtain the previous formula.

$$Tr[\tau^j \tau^k] = -2\delta^{ij} \tag{12.72}$$

Recall that we are already working under the assumption that our lattice is highly refined, that we are close to the continuum limit. Under this assumption,

$$\rho_e(x) = 1 + O(\mu) \tag{12.73}$$

where the μ is as defined above equation (12.19). Thus, we put in a $h(\rho_e(x))$ and a $h^{-1}(\rho_e(x))$ on both sides of $E^c_k \tau^k$ in the previous equation, we get,

$$\begin{aligned} \int_{S_{e_i}} d^2\sigma n_a E^a_j(\sigma) &= -\frac{1}{4} Tr \left[\tau^j \oint_{S_e} \varepsilon_{abc} \, dx^a \wedge dx^b h(\rho_e(\vec{x})) E^c_k(\vec{x}) \tau^k h^{-1}(\rho_e(\vec{x})) \right] \\ &= \frac{\beta}{2} P^j(e_i) + O(\mu^3) \end{aligned} \tag{12.74}$$

where, we have obtained the last equation from (12.19). This formula holds in the continuum limit. This implies that the lattice discretisation of area of S is given by

$$Ar(S) = \sum_{i=1}^{n} \frac{\beta}{2} \sqrt{P^j(e_i) P^j(e_i)} \tag{12.75}$$

After quantisation,

$$\hat{Ar}(S) = \sum_{i=1}^{n} \frac{\beta}{2} \sqrt{\hat{P}^j(e_i) \hat{P}^j(e_i)} \tag{12.76}$$

$$\begin{aligned} \hat{Ar}(S) &= \frac{\beta l_p^2}{4} \sum_{i=1}^{n} \sqrt{-\hat{R}^j_{e_i} \hat{R}^j_{e_i}} \\ &= \frac{\beta l_p^2}{2} \sum_{i=1}^{n} \sqrt{\hat{J} \cdot \hat{J}} \end{aligned} \tag{12.77}$$

where in the second step we have used the relation $-i\frac{\hat{R}^j}{2} \equiv \hat{J}^j$. Now, $\hat{J}\cdot\hat{J} = \hat{J}^2$ and the eigenvalue of the operator $\hat{J}^2$ is j(j+1). Thus, this formula directly gives us the eigenvalue of the area operator. This means that the spin network states are eigen vectors of the area operator because spin-network states have fixed j.

$$\hat{A}r\, T_{\gamma,\vec{j},\vec{i}} = \left(\sum_{e\cap S\neq\phi} \frac{\beta l_p^2}{2} \sum_{i=1}^{n} \sqrt{j_e(j_e+1)} \right) T_{\gamma,\vec{j},\vec{i}} \tag{12.78}$$

This is the eigen vector equation of the area operator.

The area operator acts on gauge invariant states to give gauge invariant states, because, the operator $\sqrt{\hat{P}^j(e_i)\hat{P}^j(e_i)}$ is gauge invariant. This is because, suppose you rotate the index j on both the $\hat{P}^j$'s, it is invariant. This is just an SO(3) inner product in 2 j's, that is in both the P's. Hence, the area operator is gauge invariant, SU(2) gauge invariant.

Another key point about the area operator is that it has a discrete spectrum, since j_e takes discrete values. This means our geometry is fundamentally discrete. Thus quantum Riemannian geometry is fundamentally discrete.

Why we don't see the discreteness in the area as much as we see it in spin is because, in spin, the gap between the eigenvalues is of order 1, but in area, the gap between the eigenvalues is of order l_p^2, which is a very small quantity, far beyond current values of observation. That's why you don't really see this discreteness in any current experiments. The discreteness of the geometry is quantum in nature, which is why it is not visible in the classical world.

But this is one of the most important results of loop quantum gravity, that geometry is fundamentally discrete.

The spin network wave function/states diagonalise the area operator. The reason is just because the flux operator or the right invariant vector field is an analog of (the spin operator) the angular momentum operator. The angular momentum operator and the right vector field have the same lie algebra and representation theory tells us that they are, in fact, the same.

Volume Operator

Let's consider a 3-dim region R in our spatial space, $R \subset \Sigma$. We discretise this volume of R, again, using a lattice. We can consider this volume to be made of smaller volumes concentrated at each vertex. We can distribute this volume at each vertex and the discretisation is just the dual of the vertex (the volume of the box that is dual to the vertex) for all the vertices inside the volume.

The 3-dim region R whose volume we are trying to measure is shown in black, the lattice is shown in blue and the dual lattice that we use to approximate the region R is shown in red, in the figure below.

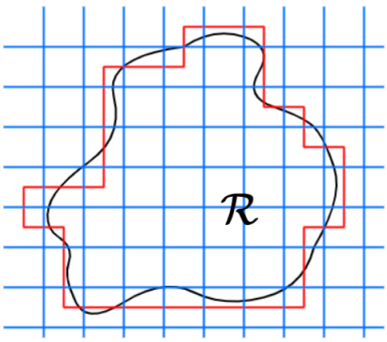

Figure 16: Approximation of a volume V

$$V(R) \simeq \sum_{v \in R} V_v \tag{12.79}$$

where V_v is the volume of the box dual to vertex V.

$$\begin{aligned} V(R) &= \int_R d^3x \sqrt{det(h)} \\ &= \int_R d^3x \sqrt{det(E_i^a)} \\ &= \int_R d^3x \sqrt{\left|\frac{1}{6}\varepsilon_{abc}\varepsilon^{ijk}E_i^a E_j^b E_k^c\right|} \end{aligned} \tag{12.80}$$

where you obtain the second equation from

$$\sqrt{det(h)} = \sqrt{det(E_i^a)} \tag{12.81}$$

Now, we have

$$V_v = \int_{B_v} d^3x \sqrt{\left|\frac{1}{6}\varepsilon_{abc}\varepsilon^{ijk}E_i^a E_j^b E_k^c\right|} \tag{12.82}$$

where B_v is the box dual to the vertex v.

Recall that our vertices were 6-valent, had three incoming edges and three outgoing edges labelled as in the figure 16. But, for the sake of convenience in our discussion of the volume operator, we relabel the edges as follows.

We orient all the edges to be outgoing and we focus on the vertex v. When we flip the edges, we have to change the holonomy along the edge into its inverse.

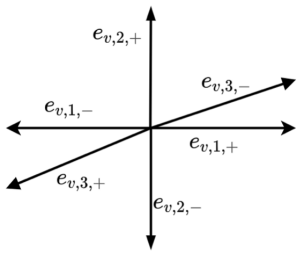

Figure 17: The relabelling of the vertices

The relation between the two labelling systems is given by,

$$e_{v,I,+} = e_I(v) \quad ; \quad e_{v,I,-} = e_I^{-1}(v - \hat{I}) \tag{12.83}$$

Now we give the discrete result

$$V(R) = \sum_{v \in R} V_v \text{ where } V_v = \sqrt{|Q_v|} \tag{12.84}$$

where we define Q_v as

$$Q_v = \beta^3 \varepsilon_{ijk} \left(\frac{P^i(e_{v,1,+}) - P^i(e_{v,1,-})}{4} \right) \left(\frac{P^j(e_{v,2,+}) - P^j(e_{v,2,-})}{4} \right) \times \left(\frac{P^k(e_{v,3,+}) - P^k(e_{v,3,-})}{4} \right) \tag{12.85}$$

That is, if we define

$$X_I^i(v) = P^i(e_{v,I,+}) - P^i(e_{v,I,-}) \tag{12.86}$$

then,

$$Q_v = \beta^3 \varepsilon_{ijk} \frac{X_1^i(v)}{4} \frac{X_2^j(v)}{4} \frac{X_3^k(v)}{4} \tag{12.87}$$

We have,

$$V(R) = \sum_{v \in R} \sqrt{|Q_v|} + O(\mu^4) \tag{12.88}$$

This formula is valid only uptil the order μ^3.

We can show that in the continuum limit, $\sum \sqrt{|Q_v|}$ converges to the volume. This can be proved using,

$$P^i(e_{v,I,-}) = \frac{1}{2} Tr[\tau^i h^{-1}(e_{v-\hat{I},I,+}) P^j(e_{v-\hat{I},I,+}) \tau^j h(e_{v-\hat{I},I,+})] \tag{12.89}$$

The continuum limit results is given by,

$$V_v = \mu^3 \sqrt{det(E_i^a)} + O(\mu^4) \tag{12.90}$$

This can be obtained by using the continuum limit of flux, given by,

$$P^j(e_{v,I,s}) = \frac{\mu^2 s}{\beta} E_I^j(v) + O(\mu^4) \text{ where } s = 1 \text{ and } I = 1,2,3 \tag{12.91}$$

On quantisation, we obtain,

$$\hat{V}(R) = \sum_{v \in R} \hat{V}_v \text{ where } \hat{V}_v = \sqrt{|\hat{Q}_v|} \tag{12.92}$$

where

$$\hat{Q}_v = \beta^3 \varepsilon_{ijk} \frac{\hat{X}_1^i(v)}{4} \frac{\hat{X}_2^j(v)}{4} \frac{\hat{X}_3^k(v)}{4} \quad \text{and} \quad \hat{X}_I^i(v) = \hat{P}^i(e_{v,I,+}) - \hat{P}^i(e_{v,I,-}) \tag{12.93}$$

The square root of the absolute value of an operator in Quantum Mechanics is defined as

$$\sqrt{\hat{Q}} = (Q^2)^{\frac{1}{4}} \tag{12.94}$$

First, you take the square of the Q operator, and then you have to diagnolise the Q operator, take its eigenvalues, and then take the fourth root of those values.

$\hat{Q}$ is self adjoint because $\hat{P}_j$ is self-adjoint. Self adjoint matrices can be diagonalised. The volume operator, $\hat{V}$ is self-adjoint, because $\hat{Q}$ is self-adjoint (An operator to the power $\frac{1}{n}$, in Quantum Mechanics is defined for self-adjoint operators. Since you can always diagonalise a self-adjoint operator, all you have to do is just take the eigenvalue and take it to the power $\frac{1}{n}$).

Since $\hat{V}$ is self adjoint and diagonalisable, we have its eigen-vectors and eigen-values as follows:

$$\hat{V}(R) T_{\gamma,\vec{j},\vec{i}} = \sum_v \lambda_v(\vec{j},\vec{i}) T_{\gamma,\vec{j},\vec{i}} \tag{12.95}$$

We find that the spin-network states diagonalise the volume operator.

The volume operator is actually the sum of volume operators at individual vertices, and the spin network basis diagonalises each of these individual volume operators simultaneously. (because the volume operator acting at different vertices are mutually commuting)

The eigenvalue obtained from the action of the volume operator on the spin network basis depends on both the spins and the intertwiners. The eigenvalues of the area operator only depend on the spins. (The area operator is also self-adjoint and is also diagonalised by the spin-network basis)

λ_v is discrete spectrum. (The expression for in terms of j and i is complicated). The eigenvalues of the volume operator, the $\lambda_v(\vec{j},\vec{i})$'s can be computed numerically, but there is no simple analytic formula.

The volume operator gives rise to some complications while trying to compute the dynamics of Loop Quantum Gravity, because its eigenvalues are complicated. This is why a lot of analytic computations of Loop Quantum Gravity are complicated.

The same property holds for the eigenvectors of the volume operator. Saying that the eigenvectors are spin-network states involves a choice of intertwiners. Previously, when we defined spin-network states, we used an arbitrary choice of basis for intertwiner states, labelled by i. But diagonalising the volume operator actually relates to a specific choice of intertwiner basis labels. But how to obtain this specific choice? You can definitely compute it numerically. The way to do this computation numerically is : If you fix a spin, on an edge, the volume operator on the vertex, relates to a rotation in the intertwiner space because you fix all the edge spins and then the volume operator acting on the

intertwiner space gives a rotation in the intertwiner space, and then you find the basis in the intertwiner space, the specific basis that diagnolises the volume operator (because volume operators act on the intertwiner space). But, to really do this computation, you have to do case by case, first fix the numbers of spin and then compute numerically, the basis. There is no one who knows how to do this computation analytically by taking generic values of the spin.

The minimum eigenvalue of the area operator is 0, which occurs when $j = 0$ in $j(j = 1)$. Similarly, the minimum eigenvalue of the volume operator is 0, $\lambda_v = 0$. This occurs when $\vec{j} = 0$. This means you have no volume either. The j=0 is the state, this state has no area either. The $j = 0$ is what we call the vacuum state. It is the state with "no geometry".

In Loop Quantum Gravity, both area and volume operators are self-adjoint and have a discrete spectrum. This indicates that quantum geometry, unlike classical geometry, is fundamentally discrete. It is discrete at the fundamental length scale, the Planck Scale, which is why this discreteness is not seen macroscopically (that is, the gaps in area and volume are of the order of l_p^2 and l_p^2 respectively.) Macroscopically, l_p is negligible and you recover smooth geometry.

Quantum geometry = spin network states. Here you have the network states and the classical geometric quantities area and volume are quantised as operators. (just like Quantum Mechanics. Classically, you have energy, momentum and so on, but in Quantum Mechanics, they are quantised operators that act on states in the Hilbert space)

Lets consider a cubic lattice. Our spin network states are defined on a cubic lattice. The edges are labelled by spins j_e and the vertices are labelled by intertwiners i_v.

For any spin network state, every edge has a spin $j_e \in \frac{\mathbb{Z}}{2}$. j_e is actually the quanta of area, because spin labels the area eigenvalues. If S_e was the surface dual to edge e, then

$$Ar(S_e) \sim \sqrt{j_e(j_e+1)} \tag{12.96}$$

And, for every vertex V, we have an intertwiner i_v , which is the quanta of volume.

Quantum geometries only exist on edges and vertices. They are distributional and non-smooth.

13 Intertwiners

13.1 Introduction

Recall that below equation (12.50), we defined the intertwiner space for a 6-valent intertwiner as

$$Inv(V_{j_1} \otimes ... \otimes V_{j_6}) \tag{13.1}$$

Thus the intertwiner space for an N-valent intertwiner is given by

$$Inv(V_{j_1} \otimes ... \otimes V_{j_N}) = \mathscr{H}_{j_1,...j_N} \tag{13.2}$$

which we denote as $\mathscr{H}_{j_1,\dots j_N}$. Now, let's denote the space of N-valent intertwiners with fixed total area $J = \sum_i j_i$ as $\mathscr{H}_N^{(J)}$. Thus,

$$\mathscr{H}_N^{(J)} = \bigoplus_{\sum_i j_i = J} \mathscr{H}_{j_1,\dots j_N} \tag{13.3}$$

Thus the total space of all N-valent intertwiners is given by a sum over all values of J [11].

$$\mathscr{H}_N = \bigoplus_{J\in\mathbb{N}} \mathscr{H}_N^{(J)} = \bigoplus_{\{j_i\}} \mathscr{H}_{j_1,\dots j_N} \tag{13.4}$$

13.2 Spinorial Formulation of LQG

Our aim is to construct the coherent states of a polyhedron. These coherent states are U(N) invariant intertwiners. In order to construct these states, we will need the spinorial formulation of the Loop Quantum Gravity phase space.

In the spinorial formulation of Loop Quantum Gravity, we associate, a spinor with each end of a single edge [10].

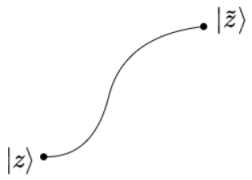

Figure 18: The spinors $|z\rangle$ and $|\tilde{z}\rangle$ of the edge e

A spinor is an element of $\mathbb{C}^2$. Therefore, the classical state space of a single edge corresponds to $\mathbb{C}^2 \times \mathbb{C}^2$.

We have

$$|z\rangle = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \quad ; \quad \langle z| = \begin{pmatrix} \bar{z}^0 & \bar{z}^1 \end{pmatrix} \tag{13.5}$$

We define a dual spinor $|z]$ belonging to the $SU(2)^*$ representation as

$$|z] = \varepsilon\,|\tilde{z}\rangle \tag{13.6}$$

We have

$$\langle z|z\rangle = [z|z] = |z^0|^2 + |z^1|^2 \tag{13.7}$$

The density matrix $\langle z|z\rangle$ is Hermitian.

We can associate a 4-dim null vector with each spinor. We write down a 2×2 matrix from the spinor $|z\rangle$ as follows,

$$|z\rangle\langle z| = \frac{1}{2}(\langle z|z\rangle\,\mathbb{1} + \vec{X}\cdot\vec{\sigma}) \tag{13.8}$$

where $\vec{X} = \langle z|\vec{\sigma}|z\rangle$.

Now, any matrix formed as a result of outer product between two vectors has determinant zero. Any 2×2 matrix with $det = 0$ can be written as $V_\mu \sigma^\mu$ where V_μ is a null vector and $\sigma^\mu = (1, \sigma^i)$. Thus, the matrix $|z\rangle\langle z|$ has $det = 0$ and we

write $|z\rangle\langle z| = X_\mu \sigma^\mu$ where $X^\mu = \frac{1}{2}\left(\langle z|z\rangle, \vec{X}\right)$ is a null vector. Thus, we say, X^μ is the null vector associated with the spinor $|z\rangle$. $\vec{X}$ is the normal to the face that is dual to the edge. The magnitude of the normal vector $\vec{X}$ gives twice the area of the face dual to the edge. Thus,

$$|\vec{X}_i| = 2A_i \tag{13.9}$$

where A_i is the area of the face dual to the i^{th} edge. Now, since $X_i\ _\mu$ is a null vector, we have, $|\vec{X}_i| = X_i^0 = \langle z_i|z_i\rangle$,

$$2A_i = \langle z_i|z_i\rangle \tag{13.10}$$

We construct a map that takes us from these pairs of spinors to the $SU(2)$ space

$$g : \mathbb{C}\times\mathbb{C} \longmapsto SU(2) \tag{13.11}$$

given by,

$$g(z,\tilde{z}) = \frac{|z\rangle[\tilde{z}| - |z]\langle\tilde{z}|}{|z|\ |\tilde{z}|} \tag{13.12}$$

where $|z| = \sqrt{\langle z|z\rangle}$. This is the norm on that arises from the fact that $z,\tilde{z}\in\mathbb{C}$. (Recall that $\forall\, z\in\mathbb{C}, |z|^2 = z\bar{z}$).

$g(z,\tilde{z}) \in SU(2)$. If $|z\rangle$ and $|\tilde{z}\rangle$ are the spinors at the end of an edge, $g(z,\tilde{z})$ corresponds to the holonomy along that edge.

$$g^\dagger(z,\tilde{z}) = \frac{|\tilde{z}]\langle z| - |\tilde{z}\rangle[z|}{|z|\ |\tilde{z}|} \tag{13.13}$$

Also, a holonomy parallel transports a spinor from one end of the edge to the other. Thus, we have a relation between $|z\rangle$, $|\tilde{z}\rangle$ and $g(z,\tilde{z})$, given by,

$$g(z,\tilde{z})\frac{|\tilde{z}\rangle}{|\tilde{z}|} = -\frac{|z]}{|z|} \tag{13.14}$$

$$g^\dagger(z,\tilde{z})\frac{|z\rangle}{|z|} = -\frac{|\tilde{z}]}{|\tilde{z}|} \tag{13.15}$$

When you have a local SU(2) gauge field h, g transforms as,

$$g \longrightarrow h(s)\cdot g\cdot h^{-1}(t) \tag{13.16}$$

where s and t denote the source and target of the edge along which g is defined. In the spinor formalism, under a local gauge transformation,

$$\left(|z\rangle, |\tilde{z}\rangle\right) \longrightarrow \left(h(s)|z\rangle, h(t)|\tilde{z}\rangle\right) \tag{13.17}$$

Thus,

$$g(z,\tilde{z}) \longrightarrow g(h(s)|z\rangle, h(t)|\tilde{z}\rangle) = h(s)g(z,\tilde{z})h^{-1}(t) \tag{13.18}$$

13.3 Area Matching Constraint

Recall that area corresponds to the length of the normal vector and that in this spinorial picture, the normal vector is $\vec{X}$. Thus, the area matching constraint is given by [12],

$$M = \langle z|z\rangle - \langle\tilde{z}|\tilde{z}\rangle = 0 \tag{13.19}$$

Now, we want to write down the spinors as elements of a phase space, that is we want to put a symplectic structure on $\mathbb{C}^2 \times \mathbb{C}^2$. Consider $|x\rangle \in \mathbb{C}$, such that

$$|x\rangle = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{13.20}$$

If we consider x^0, x^1 as one "position" variables, then that is their complex conjugates, $(x^0)^*, (x^1)^*$ are the "momentum" variables. The Poisson bracket structure of this phase space is given by,

$$\{(x^i)^*, x^j\} = i\delta^{ij} \tag{13.21}$$

Thus $\mathbb{C}^4$ is a phase space.

Now, the area matching constraint is a function on the phase space. Thus, we can calculate the Poisson bracket between the constraint and the spinors

$$\{M, |z\rangle\} = i\,|z\rangle \quad ; \quad \{M, |\tilde{z}\rangle\} = -i\,|\tilde{z}\rangle^* \tag{13.22}$$

The area matching constraint generates a U(1) gauge transformation on the phase space. It generates the transformation

$$|z\rangle \longrightarrow e^{i\phi}\,|z\rangle \quad ; \quad |\tilde{z}\rangle \longrightarrow e^{-i\phi}\,|\tilde{z}\rangle \tag{13.23}$$

This $g(z,\tilde{z})$ is invariant under this U(1) action and so is $X = \vec{X}\cdot\vec{\sigma}$. $g \in SU(2)$ and $X \in \mathfrak{su}(2)$ lie algebra.

14 U(N) Invariant Intertwiners

14.1 Geometry of Polyhedra

At a vertex V, the dual faces of the edges form a polyhedra. For this to happen it has to satisfy the closure constraint. For a classical polygon, the constraint is that the face normals $\vec{n}_i$ must satisfy

$$\sum_{i=1}^{N} \vec{n}_i = 0 \tag{14.1}$$

For a quantum polyhedron, for the spin operators acting on the N-faces the closure condition becomes,

$$\sum_{i=1}^{N} \vec{J}_i\,|\psi_\Gamma\rangle = 0 \tag{14.2}$$

where

$$\sum_{i=1}^{N} \vec{J}_i = \sum_{i=1}^{N} (\mathbb{1}_1 \otimes \ldots \otimes \vec{J}_i \ldots \otimes \mathbb{1}_N) \tag{14.3}$$

This constraint is identical to the Gauss constraint.

Recall that in the spinorial formulation of LQG, vectors normal to the faces dual to the edges are given by $\vec{X} = \langle z|\vec{\sigma}|z\rangle$, where $|z\rangle$ is the spinor associated with that edge. Thus, the closure constraint can be written in terms of the spinors as

$$\sum_i \vec{X}_i = 0 \tag{14.4}$$

Now, we want to construct the semi-classical states of quantum geometry. These are the least uncertainity states, that is, the states that saturate the bound on the uncertainty of a state given by the Hisenberg Uncertainty Principle, $\Delta x \Delta p \geq \frac{\hbar}{2}$ where $\Delta x = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$. These are also called the coherent states.

Now, we study 4-valent intertwiners in detail. If we take all 4 edges to be spin 1/2, our Hilbert space becomes

$$\bigotimes_{k=1}^{4} \mathscr{H}_{\frac{1}{2}} = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \tag{14.5}$$

The $0 \oplus 0$ part is the scalar part. It is invariant under SU(2) transformations. Now we can write,

$$I_{j_1 \ldots j_4} = Inv \left[\bigotimes_{k=1}^{4} \mathscr{H}_{j_i} \right] \tag{14.6}$$

Now, associated with each $\mathscr{H}_{j_i}$, you have its own set of SU(2) operator and its own Casamir operator $\vec{J}^2$. The quantum geometry, in case of $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$ is described by the invariant subspace given by $0 \oplus 0$, it is two dimensional.

Now we consider a classical tetrahedron (which is the simplest 3-d solid because a cube has 6 faces and hence has 6 face normals, while the tetrahedron has only 4). It is given by the 4 face normals whose magnitude gives twice the area of the face

$$|\vec{n}_i| = \sqrt{\vec{n}_i \cdot \vec{n}_i} = 2A_i \tag{14.7}$$

The dihedral angle θ_{ij} is the angle between the i^{th} and the j^{th} face.

$$\vec{n}_i \cdot \vec{n}_i = |\vec{n}_i| \; |\vec{n}_j| \; cos(\theta_{ij}) \tag{14.8}$$

Apart from the closure constraint, we have a constraint for the dihedral angles,

$$\vec{n}_3 \cdot \vec{n}_4 = \vec{n}_1 \cdot \vec{n}_2 + A_1 + A_2 - A_3 - A_4 \tag{14.9}$$

An equivalent way to describe a tetrahedron instead of using face normals, is by the edge vectors e_1, e_2 and e_3. Any arbitrary set of 3 vectors can define a tetrahedron. Therefore, total number of degrees of freedom = 9. But to define the tetrahedron, uniquely, we need to mod out SO(3) rotations (because, geometrically, we do not want to distinguish between rotated tetrahedrons). Therefore, total no. of physical degrees of freedom = 9-3 = 6.

We take these 6 numbers to be

$$\{\vec{e}_1 \cdot \vec{e}_1 \; ; \; \vec{e}_2 \cdot \vec{e}_2 \; ; \; \vec{e}_3 \cdot \vec{e}_3 \; ; \; \vec{e}_1 \cdot \vec{e}_2 \; ; \; \vec{e}_2 \cdot \vec{e}_3 \; ; \; \vec{e}_1 \cdot \vec{e}_3\}$$

that is, the lengths of these vectors and the angle between pairs of these vectors.

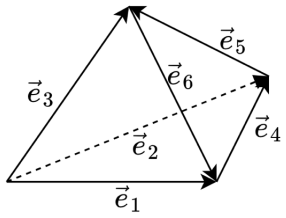

Figure 19: The tetrahedron formed by e_1, e_2 and e_3

Now, we have 6 edge vectors (we give an arbitrary orientations to the other 3 edges) and 4 face normals.

The 6 edge vectors are related to each other as,

$$\vec{n}_1 = -\vec{e}_1 \times \vec{e}_2 \tag{14.10}$$

The negative sign arises because, we want the normal to face outward, but as one can see from the figure above, $\vec{e}_1 \times \vec{e}_2$ faces towards the interior of the tetrahedron. Similarly, we have

$$\vec{n}_2 = -\vec{e}_2 \times \vec{e}_3 \text{ and } \vec{n}_3 = -\vec{e}_3 \times \vec{e}_1 \tag{14.11}$$

The closure constraint can be obtained by

$$\begin{aligned} \vec{n}_4 &= \vec{e}_4 \times \vec{e}_5 = \vec{e}_5 \times \vec{e}_6 \\ &= (\vec{e}_2 - \vec{e}_1) \times (\vec{e}_3 - \vec{e}_2) \\ &= \vec{e}_2 \times \vec{e}_3 - \vec{e}_1 \times \vec{e}_2 + \vec{e}_1 \times \vec{e}_2 \\ &= -\vec{n}_2 - \vec{n}_3 - \vec{n}_1 \end{aligned} \tag{14.12}$$

Our main aim is to produce the coherent states of a quantum tetrahedron (polyhedron). For a classical tetrahedron, we have six geometrical variables, given by,
1) Four face areas, A_1, A_2, A_3 and $A4$ and
2) Two of the dihedral angles, say θ_{12}, θ_{34}

But in the quantum case, we cannot take all six variables since (they are not commuting) they have non-zero commutation relations. So what we need is a complete set of commuting operators (CSCO) to specify the quantum state. So we take only 4 out of 5.

The commutation relations of these 6 operators are,

$$[\hat{A}_i, \hat{A}_j] = 0 \tag{14.13}$$

$$[\hat{\theta}_{12}, \hat{\theta}_{34}] \neq 0 \tag{14.14}$$

This is because $\hat{\theta}_{34}$ is defined from j_3 and j_4 which are not independent from j_1 and j_2 (which define $\hat{\theta}_{12}$), because of the closure constraint.

$$[\hat{\theta}_{12}, \hat{\theta}_{23}] \neq 0 \tag{14.15}$$

$$[\hat{\theta}_{12}, \hat{A}_i] = 0 \tag{14.16}$$

Hence our CSCO will be $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4, \hat{\theta}_{12}$.

14.2 Schwinger Boson Representaion

Our aim is to construct the coherent states of a polyhedron. These coherent states are U(N) invariant intertwiners. In order to construct these states, we will need the spinorial formulation of the Loop Quantum Gravity phase space. This formulation is the classical analog of the Schwinger-Boson picture.

The basic (geometric) kinematic operators of Loop Quantum Gravity are the angular momentum operators associated with any edge. Thus, we have a set of operators $\vec{J_i}$ where $i = 1, 2, ...n_e$.

In a harmonic oscillator algebra, $J_- J_+$ is the number operator. Now, lets take two copies of a harmonic oscillator, given by $a, a^\dagger, b, b^\dagger$ with commutation relations,

$$[a, a^\dagger] = [b, b^\dagger] = 1 \text{ and } [a, b] = 0 \tag{14.17}$$

The $[a,b] = 0 \implies$ the harmonic oscillators are not coupled. These operators act on bosonic states (because for femionic states we have the relation $\{a, a^\dagger\} = 1$).

In this system, we define,

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) = \frac{1}{2}(n_a - n_b) \tag{14.18}$$

where n_a is the number operator of the a harmonic oscillator a. The eigen states of the system are labelled by both the number operators, $|n_a, n_b\rangle$. The action of the number operators on the state are,

$$a^\dagger a\,|n_a, n_b\rangle = n_a\,|n_a, n_b\rangle \quad ; \quad b^\dagger b\,|n_a, n_b\rangle = n_b\,|n_a, n_b\rangle \tag{14.19}$$

$J_+ = J_-^\dagger$ and we have $J_+ = a^\dagger b$ and $J_- = b^\dagger a$. Thus the J_+ and J_- are the hopping operators. (one quanta of a hops to one quanta of b and vice versa). We also define a total number operator

$$E = \frac{1}{2}(a^\dagger a - b^\dagger b) \tag{14.20}$$

Using J_+ and J_-, we can construct J_x and J_y. Thus we have a complete set of SU(2) algebra. Thus, we have the correspondence

$$|j, m\rangle \longleftrightarrow |n_a, n_b\rangle \tag{14.21}$$

where $j = \frac{1}{2}(n_a + n_b)$ and $m = \frac{1}{2}(n_a - n_b)$. j is the total angular momentum. It can also be thought of as the kinetic energy associated with that edge. As long as j is fixed, we are dealing with a finite state space. (a space which has a finite number of states)

$$\hat{J}^2\,|j, m\rangle\,\hat{E}(\hat{E} + \mathbb{1}) \tag{14.22}$$

Equipped with this knowledge, we can study the algebra associated with a polyhedron.

$$|n_a, n_b\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}}{\sqrt{n_a!\, n_b!}}\,|00\rangle \tag{14.23}$$

Now we know how to construct the coherent states of a harmonic oscillator. (Recall that the coherent states of a harmonic oscillator are the eigen states of the anhilation operator.)

This is the algebra of the intertwiner state space, the algebra of the polyhedron. Consider a N-valent vertex, with each edge labelled by a representation of SU(2). These operators defined on each edge give exactly the corresponding angular momentum state space for the degrees of freedom living on the edge.

14.3 Creation and Annhilation Operators

Now, we define [13],

$$E_{ij} = (a_i^\dagger a_j - b_i^\dagger b_j) \tag{14.24}$$

from which we have

$$E_i = E_{ii} = \frac{1}{2}(a_i^\dagger a_i - b_i^\dagger b_i) \tag{14.25}$$

Note that the factor of 1/2 does not occur in the definition of E_{ij} like it does in that of E_i. We have

$$E_{ij}^\dagger = E_{ji} \tag{14.26}$$

E_{ij} increases the spin of j_i by $\frac{1}{2}$ and decreases the spin of j_j by $\frac{1}{2}$. Thus, it keeps the total area of the polyhedron $J = \sum_e j_e$ constant. This is also like a hopping term, except that in this case, the hopping of the quanta occurs from one edge to an adjacent edge. Also note that quanta of a (or b) hop from one edge to another but they never hop from a to b or b to a.

These E_{ij} operators have a very nice property. Say, we consider the SU(2) symmetry of the polyhedral state space. Its generators are $\vec{J}_i$. $\vec{J} = \sum_{i=1}^{n_e} \vec{J}_i$ is the generator of the SU(2) symmetry group of the entire polyhedron. Now, $\forall\ i, j$, we have,

$$[\vec{J}, E_{ij}] = 0 \tag{14.27}$$

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \tag{14.28}$$

Equation (14.28) is a particular set of relations that defines a certain kind of lie algebra. U(N) is the group whose lie algebra is defined by the set of relations in (14.28). And the fact that all of the E_{ij}'s commute with the SU(2) generator J, tells us that.

Since all the E_{ij}'s commute with J, their eigenvalues can be used to label the intertwiner states. This is a gauge invariant basis (because its invaiant under SU(2) rotations).

Our polyhedral state space has U(N) symmetry. That's where the term U(N) intertwiners comes from.

The unitary group U(N). The generators of U(N) are skew Hermitian. Now, $det(U)$ is not necessarily 1. Thus $U(N)$ is not a sub group of $SL(N,\mathbb{C})$

$$\text{Total number of generators of } U(N) = N^2$$

$$\text{Number of diagonal generators } = N$$

$$\frac{N(N-1)}{2} = \text{Number of real off-diagonal generators}$$

$$= \text{Number of imaginary off-diagonal generators}$$

Let us denote the generators of U(N) as $E_{ij}\ \forall\ i,j \in \{1,2,...N\}$ and E_i as the diagonal generator. The commutator of this lie algebra is given by (14.28).

Now, we define [11],

$$\forall\ i \neq j,\ F_{ij} = a_i b_j - a_j b_i \tag{14.29}$$

$$\implies F_{ij}^\dagger = -F_{ji} \tag{14.30}$$

F_{ij} decreases both j_i and j_j by a $\frac{1}{2}$. Thus, under the action of the F operator, the total area $J = \sum_e j_e$ decreases by 1. And the $F^\dagger$ operator increases the total area by 1.

Thus the F and the $F^\dagger$ operators can be thought of as the creation and the annihilation operators that induce the Hilbert space $\mathscr{H}_N = \bigoplus_{J\in\mathbb{N}} \mathscr{H}_N^{(J)}$ that we introduced in equation (13.4) with a Fock space structure. The operators E_{ij} take states from each subspace $\mathscr{H}_N^{(J)}$ to states in the subspace itself, where as the F and the $F^\dagger$ operators take states from one subspace to another with a different total area J.

14.4 Construction of Coherent States

Let's consider an N-valent intertwiner, one whose state is specified by the N spinors $|z_i\rangle$, $i = 1,2,...N$. We can define an $SL(2,\mathbb{C})$ action on these spinors, given by

$$\text{for some } \lambda \in SL(2,\mathbb{C}),\ \text{we have } |z_i\rangle \longrightarrow \lambda\,|z_i\rangle \tag{14.31}$$

We use these N spinors $\{|z_i\rangle\}$ to construct an $N \times N$ antisymmetric matrix that is invariant under the transformation in (14.31), given by,

$$\mathbf{z}_{ij} = [z_i\,|z_j\rangle = z_i^0 z_j^1 - z_j^0 z_i^1 \tag{14.32}$$

Now, we define a creation operator $F_{\mathbf{z}}^\dagger$ as follows,

$$F_{\mathbf{z}}^\dagger = \frac{1}{2}\sum \mathbf{z}_{ji} F_{ij}^\dagger \tag{14.33}$$

Note that we use the matrix $\mathbf{z}_{ji}$ in the definition of $F_{\mathbf{z}}^\dagger$ and not $\mathbf{z}_{ij}$ like we defined in equation (14.32).

We denote states in the Fock space as $|\psi)$ with round brackets instead of angular brackets to avoid confusion with the spinors. Now, for a given integer J $\in \mathbb{N}$, we define a coherent state of total area J as,

$$|J, z_i) \equiv \frac{1}{\sqrt{(J+1)}} \frac{(F_{\mathbf{z}}^{\dagger})^J}{J!} |0) \tag{14.34}$$

where $|0)$ is the Fock vacuum [11]. In this paper, it has also been proved that the states defined above in equation (14.34) are indeed coherent.

The coherent states we have just defined are labeled by the n spinors $\{z_i\}, i = 1, 2, ...n$. Recall that we defined an $SL(2,\mathbb{C})$ action on the spinors, given by (14.31). Now, we define a $GL(2,\mathbb{C})$ action on the spinors as,

$$\text{for some } \lambda \in GL(2,\mathbb{C}), \text{ we have } |z_i\rangle \longrightarrow \lambda\, |z_i\rangle \tag{14.35}$$

Under this action, the matrix $\mathbf{z}_{ji}$ transforms as $\mathbf{z}_{ji} \longrightarrow det(\lambda)\mathbf{z}_{ji}$ and the $F_{\mathbf{z}}^{\dagger}$ operator transforms as $F_{\mathbf{z}}^{\dagger} \longrightarrow det(\lambda)F_{\mathbf{z}}^{\dagger}$. This implies that our coherent states transform as homogeneous functions of degree J,

$$|J, \lambda z_i) \longrightarrow det(\lambda)^J |J, z_i) \tag{14.36}$$

Since states that differ by a constant represent the same physical state, our coherent states are labelled uniquely by points in $\mathbb{C}^{2N}/GL(2,\mathbb{C})$. This is exactly the Grassmannian that we defined in (6.3). Thus the space of U(N) invariant intertwiners, the space of unique labels of the Freidel-Livine coherent states, is the Grassmannian $G_{N,2}$, which is exactly what we have shown to be the kinematic space of the scattering of N massless particles.

15 Correspondence

The N spinors in the Friedel-Livine coherent states corresponds to the N spinors in the spinor helicity picture, denoting the kinematical data (the momenta) of the scattering of N massless particles. We now try to establish this correspondence on more solid, mathematical grounds.

Since $|z_i\rangle$ is an $SU_L(2)$ spinor, we have the first correspondence, and from that, we have the next one.

$$\begin{aligned} |p_i] &\longleftrightarrow |z_i\rangle \\ [p_i| &\longleftrightarrow -[z_i| \end{aligned} \tag{15.1}$$

If we impose the constraint that momentum is real, then from equation (4.23) we have

$$\begin{aligned} |p_i\rangle &\longleftrightarrow -|z_i] \\ \langle p_i| &\longleftrightarrow \langle z_i| \end{aligned} \tag{15.2}$$

Recall from equation (13.8) that

$$|z_i\rangle \langle z_i| = \frac{1}{2}(\langle z_i|z_i\rangle \mathbb{1} + \vec{X}_i \cdot \vec{\sigma})$$

where $\vec{X}_i = \langle z_i|\vec{\sigma}|z_i\rangle$. Now, from our correspondence relations in (15.1) and (15.2), we have

$$|z_i\rangle \langle z_i| \longleftrightarrow -|p_i] \langle p_i| \tag{15.3}$$

Note that the underlying assumption we make when we make this correspondence is that the momenta are real.

Recall that we write,

$$|z_i\rangle\langle z_i| = (X_{i\ \mu}\sigma^{\mu}) \tag{15.4}$$

where we define $X_{i\ \mu} = \frac{1}{2}\left(\langle z_i|z_i\rangle, \vec{X}_i\right)$. Now, recall from equations (4.3) and (4.5) that

$$p_{i\ \ ab} = (p_{i\ \mu}\sigma^{\mu}) = -|p_i]_a\langle p_i|_b \tag{15.5}$$

Thus the correspondence in (15.3) implies the correspondence $X_{i\ \mu} \longleftrightarrow p_{i\ \mu}$, which gives us the correspondence,

$$X_i^{\mu} \longleftrightarrow p_i^{\mu} \tag{15.6}$$

Note that such a correspondence may hold for any set N null vectors $X_{i\ \mu}$ and a set of null momenta $p_{i\ \mu}$ and that such a correspondence between null vectors can lead to the correspondence given in equations (15.1) and (15.2). But the spinor helicity variables in scattering amplitudes and the spinors in the spinorial formulation of LQG are subject to further constraints.

We know that the spinor helicity variables make the constraint that the momenta are null manifest. But, the momentum of scattering amplitudes are also subject to the constraint of momentum conservation. Recall that this constraint is imposed on the spinor helicity variables in the form of equation (4.16), given by, $\sum_i |i\rangle[i| = 0$. Recall that we impose the closure constraint on the spinors of the N edges of the intertwiner during the construction of the Freidel Livine coherent states. Recall from (14.4) that the closure constraint is given by, $\sum_i \vec{X}_i = 0$. This condition corresponds to the condition that the spatial components of momentum are conserved, by the correspondence given in (15.6). Hence, the momentum conservation constraint that we impose on the spinor helicity variables is a stronger condition than the closure constraint that we impose on the spinor variables in the intertwiner coherent states.

Now, by the correspondence given in (15.6), we have,

$$\begin{aligned} p_i^0 &\longleftrightarrow \frac{1}{2}\langle z_i|z_i\rangle \\ \vec{p}_i &\longleftrightarrow \frac{1}{2}\vec{X}_i \end{aligned} \tag{15.7}$$

where the first relation is given with a modulus because the momentum conservation constraint that we impose on the spinor helicity variables is a stronger condition than the closure constraint that we impose on the spinor variables in the intertwiner coherent states. Now, $|p_i^0|$ is the positive energy of the i^{th} scattering particle. Recall that $\langle z_i|z_i\rangle = 2A_i$, where A_i is the area of the face dual to the i^{th} edge. Thus, $\exists$ a correspondence between the area of each face and the positive energies of the particles present in the scattering process. This shows that the total energy of the scattering process corresponds to the total area associated with the coherent state.

16 Conclusion

Thus, we have successfully proved that there exists a correspondence between the Spinor Helicity formalism of calculating scattering amplitudes and the spinorial formulation of LQG and shown that the Friedel-Levine coherent states

of N-valent intertwiners can be thought of as states in the kinematic space of an N-particle scattering amplitude.

Whether this correspondence is a result of a mathematical coincidence, or suggests a deeper relation between the theory of scattering amplitudes and LQG remains to be seen.

17 Moving Forward

Moving forward, this correspondence can be used to construct the momentum twistors and an amplituhedron formalism for LQG to be able to calculate scattering amplitudes in LQG. If we can obtain the graviton propagators obtained by Rovelli and others in these papers using our amplituhedron formalism for LQG, this would be an important step in the direction of establishing LQG as a valid contender for a theory of quantum gravity.

Another important consequence of our result of the area-energy correspondence is the fact that our coherent states reach their semiclassical limit as $J \longrightarrow \infty$, as area increases which seem to suggest that our particle scattering must also approach the semiclassical limit as energy increases. This seems to suggest that quantum gravity may be an infrared regulator rather than the UV regulator that common belief leads us to believe. This is a possibility that my supervisor and I are currently exploring.

Another possible avenue of research that stems from this result is a relation of this work to previous work done by my supervisor on the Complexity = Momentum correspondence. Now, quantum processes can be described as a quantum circuit and the complexity of the circuit which is defined as the number of gates in the circuit. Now, if we calculate the complexity of a scattering process, it increases with an increase in the momentum of the scattering particles. This is the well established Complexity=Momentum correspondence. Now, our area-energy correspondence seem to suggest that the scattering process associated with an intertwiner state of large total area is large. This is a possibility that my supervisor is currently exploring.

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