Conformal Field Theory
Information Loss Paradox
and String Theory

Project Report

submitted in partial fulfillment of the requirement for
the degree of

MASTER OF SCIENCE

IN

PHYSICS

by

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DECLARATION

I hereby declare that the report for the post-graduate project work entitled "Conformal Field Theory,Information Loss Paradox and String Theory" which is submitted to National Institute of Technology Karnataka Surathkal, in partial fulfillment of the requirements for the award of the Degree of Master of Science in the Department of Physics, is a bonafide report of the work carried out by me. The material contained in this report has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to certify that the project entitled “Conformal Field Theory,Information Loss Paradox and String Theory” is an authenticated record of work carried out by RAKSHIT MANDISH GHARAT, Roll No.: 196PH018 in partial fulfillment of the requirement for the award of the Degree of Master of Science in Physics which is submitted to Department of Physics, National Institute of Technology Karnataka, Surathkal, during the period 2020-2021.

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ACKNOWLEDGEMENT

I would like to express my sincere thanks to my project advisor Dr. Deepak Vaid, Assistant Professor, Department of Physics, National Institute of Technology Karnataka, Surathkal, for his timely guidance and support. I would like to thank our project coordinator Dr. N. K. Udayashankar, Department of Physics, National Institute of Technology Karnataka, Surathkal for the valuable project opportunity that he provided for me. I thank all those who have helped in the course of this project.

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ABSTRACT

The thesis aims to introduce some peculiar, mathematical as well as qualitative, concepts of three models viz., Conformal Field Theory(CFT), Hawking Information Paradox and Bosonic strings.

We start off with the first model of CFT by giving mathematical descriptions of conformal transformations. These mathematical techniques are then used to compute the generators of conformal symmetry for a conformally invariant quantum field theory, using Ward identities and operator product expansions. The generators are finally used to form the Lie algebra for CFT, also known as Virasoro algebra.

In the second model we tackle the problem of applying rules of quantum mechanics to study evaporating black holes. AdS/CFT correspondence principle is used to the case of eternal black holes, further extended to introduce Hawking process. Subsequently, a qualitative analysis of Hawking information paradox is showcased by employing use of semi-classical formula for gravitational systems. The resolution of the paradox is then reviewed by employing use of quantum extremal surfaces and islands.

In the final model we lay out a brief mathematical introduction of Bosonic strings. Starting off with action principles of such strings, we use lightcone coordinates to study the symmetries exhibited by these principles. Finally, we conclude this analysis by quantising the action principles of Bosonic strings and reviewing the proof of 26 dimensional Bosonic string theory.

LITERATURE REVIEW

The historical path of research in theoretical physics has been one leading towards unifying various models. The quest of unifying dates back to late nineteenth and twentieth centuries, starting with Maxwell unifying electricity and magnetism into a intricate framework, today known as Maxwell's laws of electromagnetism [15]. Later Einstein developed special theory of relativity unifying mechanics and electromagnetism [4]. A few years later again, Einstein formulated an elegant framework unifying relativity and gravity, termed as general theory of relativity [3]. The theories of weak and electromagnetic forces were then unified using rules of quantum mechanics [19]. A major unification was done around the same period by developing a framework describing three of the four fundamental forces of by forming gauge theories using quantum field theory(QFT). As a first model in this thesis we review a special branch of QFT termed as Conformal Field Theory(CFT) . The mathematical framework reviewing CFT has been developed as prescribed by Paul Ginsparg in the lectures given by him at Les Houches summer session in 1988[7]. The lecture notes seem to have a rather more physically intuitive content and a less mathematically rigorous approach. To add some mathematical rigour in the content we have referred to the lectures given by Tobias J. Osborne [11]. A much more mathematically rigorous analysis of CFT can be found in the book by Martin Schottenloher [20].

Continuing the quest of unification, Stephen Hawking boldly used quantum mechanics to describe a gravitational system of evaporating black holes [9]. This attempt at unification of gravity with QFT made it possible to explore many unexplored territories of black hole thermodynamics by using rules of quantum mechanics. Another major development in an attempt to unify QFT and graivty, is an intricate framework developed by Maldacena, today termed as AdS/CFT correpondence principle [13]. The duality has been widely used in almost all the regimes of theoretical physics. In the thesis, as a second model we employ the use of AdS/CFT dulaity to review the case of Eternal Black holes. The work presents a quick review of the paper published in 2003 again by Maldacena [14]. We extend the model by reviewing the case of black hole information paradox described in a paper by Almheiri et al [2]. The mathematical formulation presented in this thesis describing quantum mechanics of black holes has been referred from Lecture notes of Thomas Hartman [8]. We have also referred to some prominent textbooks

for reviewing AdS/CFT duality. [16][5]
Since past four decades or so, the most promising and leading candidate for a theory describing all four fundamental forces as a unique framework has been string theory. We review the case of the simplest strings known as Bosonic strings as the final model of this thesis. We have referred to the lectures of Shiraz Minwalla for this review. For understanding physical impacts of the formulation we referred to the lecture notes of David Tong [22]. For reviewing some literary notes on superstrings we referred to various textbooks on the same [12][17].

OBJECTIVES AND SCOPE

This thesis demonstrates some peculiar and rudimentary principles of three models viz., CFT, Quantum black holes and Bosonic strings. The main objectives are:

- To get familiar with mathematical formulation of conformal group and transformation.
- To form Lie algebra which one requires to understand the basic framework of a conformally invariant quantum field theory.
- To apply the AdS/CFT duality in the case of Eternal black holes and hence understand some fundamental properties of Hawking radiation.
- To provide a qualitative analysis of Hawking information paradox and the attempts made to resolve it.
- To understand basic action principles obeyed by Bosonic strings.
- To understand mathematical formulations one requires to canonically quantise Bosonic strings.

The study of quantum gravity poses many mathematical as well as philosophical difficulties. All the models discussed in this thesis are prominent works encompassed within the framework of quantum gravity. Though, a rudimentary approach has been followed in this thesis, it can surely help one to get introduced to the theory. The mathematical techniques demonstrated in this thesis will be helpful for one to get acquainted with some basic yet fundamental principles of quantum theory of gravity.

Contents

Chapter 1

Introduction

Physics has unraveled the mysteries of universe by presenting to us physical laws in the form of mathematical equations. Historically, the journey of physics can be described one of revealing more and more secrets of the universe as one moves more and more towards the elementary level. Human mind on the other hand, seems to have a tendency towards unification and finding a unique framework to describe seemingly disparate phenomena. The four fundamental forces viz., gravitational, electromagnetic, strong nuclear force and weak nuclear force govern the fate of the universe. One of the most prominent quests of theoretical physics has been to unify these four forces. However, the unification presents us with a lot of mathematical as well as philosophical difficulties. One has to study various models of theoretical physics to form a unique framework describing the unification of these forces; the theory thus formed will be the "Theory of Everything".

This thesis provides one with a brief review of three models viz, Conformal Field Theory (CFT), Black Hole Thermodynamics and Bosonic String theory. The main aim of the thesis is to make us understand logical, physical as well as mathematical aspects of these models. All the models contribute a lot in the quest of framing a theory of everything. Although this thesis gives a detailed description of these models, all the models themselves serve as individual giants in the branch of theoretical physics. We can say that we have just scratched the surface of these giants in the thesis. The three forces viz., electromagnetic, strong and weak nuclear force are claimed to have been unified by using the quantum field theory (QFT). Conformal field theory is a special branch of QFT. CFT can be defined as a conformally invariant QFT. The first model gives a brief introduction to such conformally invariant QFT. We introduce the mathematical aspects of conformal symmetry in section 2.2. In section 2.3 we study the aspects of conformal symmetry for the special case of two dimensions. Section 2.4 discusses the application of conformal symmetry to a classical field theory. Instead of directly quantizing this classical theory we tackle the problem in a rather indirect way by assuming that we already have a conformally invariant theory and therefore studying the constraints on such a QFT placed by conformal symmetry. This is discussed in section 2.5. Sections 2.6 to 2.9 aim to use these constraints to find the generators of the conformal symmetry. In section 2.10 we deal with the first example of a "physical" case of a CFT viz, massless free Bosons. Sections 2.11 and 2.12 finally aim to form the commutator algebra and Hilbert space for a CFT. QFT unifies the three forces in the form of gauge theories. It has been a very difficult task to form a gauge theory for gravitational force. AdS/CFT correspondence elegantly promises to connect these gauge theories and gravitational theory. Without giving a detailed description of AdS/CFT duality itself, as an extension of the first model we aim to apply the AdS/CFT duality to understand the Hawking process and hence the black hole thermodynamics concerned with the case of information paradox. We start off by briefly studying some main features of AdS spacetimes in section 3.2. Sections 3.4 to 3.6 discuss the aspects of Hawking process by first applying AdS/CFT duality to the case of Eternal black hole, further preparing a quantum state for emitted Hawking radiation quanta. Sections 3.7 to 3.9 aim to make us understand the black hole thermodynamics to a greater depth by focusing on the case of entropy. Section 3.10 discusses the origin of black hole information paradox. Sections 3.11 to 3.14 discuss different aspects of the resolution of information paradox. String theory is promised to be a healthy candidate for a theory of everything. As a last model for this theory, we aim to study the simplest case for string theory: Bosonic string theory. It serves as an important case study before one can move further to study string theory in greater depth. Section 4.2 makes us understand mathematical aspects of the action principles of Bosonic strings. Sections 4.3 and 4.4 deal with the quantization of these action principles and finally form the quantum theory for Bosonic strings. The sections briefly review the symmetries followed by these strings. We end by reviewing the proof that the Bosonic strings must quantize in far more dimensions than the three spatial dimensions that are familiar to us.

At the beginning of each model, we provide the readers with an introductory literary note explaining the need to understand the model. Before switching over to the next model we list the important features of the analysis of particular model as Concluding Remarks. Finally at the end of the thesis we conclude the analysis by exploring the inter-connections of the three models and their contributions in forming the theory of everything.

Chapter 2

Conformal Field Theory

2.1 Introducing Conformal Field Theory

In physics, field is a quantity which has a value at each point in space and time and can be represented by a number or a tensor. Now, if the system under consideration is governed by field equations pertaining to classical mechanics we term the theory as classical field theory. On the other hand if the field equations are derived using the rules of quantum mechanics, the field theory is termed as Quantum Field Theory.

As a first model in this thesis, we will be dealing with such field theories. We aim to study a type of symmetry or transformation termed as Conformal Transformation. This transformation when applied to fields whether they might be classical fields or quantum fields, will put some constraints on the system. We aim to study these constraints and the implications of such constraints on the whole field theory itself. The theory thus formed will be termed as Conformal Field Theory(CFT).

2.2 Conformal Groups and Conformal Transformations

Consider spacetime described by manifold $M = \mathbb{R}^{p,q}$ or $\mathbb{R}^d$, where $d = p + q$, and $p, q \in \mathbb{Z}_{\geq 0}$. We assign the metric to this manifold given by

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \tag{2.2.1}$$

Now suppose we apply a coordinate transformation given by,

$$x \to x' = x'(x) \text{ , with } x = (x^1, x^2, \ldots, x^p, x^{p+1}, \ldots, x^{p+q}) \tag{2.2.2}$$

then the metric will transform accordingly. The transformed metric can be given by,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') \equiv \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x') \tag{2.2.3}$$

By definition a conformal group is the subgroup of coordinate transformations that leaves the metric invariant upto a scale change. Thus under a conformal transformation the metric transformation is given by,

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.2.4}$$

where $\Omega(x) > 0$ is the (local) scale factor. Therefore a transformation which obeys the above given equation can be termed as a conformal transformation. These transformations when applied to a coordinate system will always preserve angles between two vectors v and w. The preserved angle is given by,

$$\angle\theta = \frac{g_{\mu\nu} v^\mu w^\nu}{\sqrt{(g_{\mu\nu} v^\mu w^\nu)^2}}. \tag{2.2.5}$$

Now we are ready to classify the components of conformal group defined on the manifold $(\mathbb{R}^{p,q})$. Let us denote this conformal group by $G = \mathrm{Conf}(\mathbb{R}^{p,q})$. Now consider an infinitesimal conformal transformation on the spacetime coordinates given by,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{2.2.6}$$

Under such a transformation the conformally invariant metric will be given by,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2). \tag{2.2.7}$$

The transformed metric will in turn put a constraint on the spacetime coordinates. Along with such transformation we assume that the conformally transformed metric is proportional to flat spacetime metric i.e. $g'_{\mu\nu} \propto \eta_{\mu\nu}$. These two conditions when mathematically worked out will give rise to the following equation,

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{2.2.8}$$

Such a conformal transformation will in turn give us the scale factor,

$$\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon). \tag{2.2.9}$$

The above equations will in turn give rise to a very important equation, which will help us to classify the conformal group. This equation is given by,

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu)(\partial \cdot \epsilon) = 0 \tag{2.2.10}$$

The above equation implies that ϵ should at least be differentiable three times. Also all third derivatives of ϵ are zero.

2.2.1 Classification of Infinitesimal Conformal Translations for $d > 2$

The above equations exhibit the constraints put on the spacetime coordinates. We thus deal with four types of infinitesimal transformations, , defined via ϵ, allowable in a conformal transformation: one constant, two linear, and one quadratic in spacetime coordinates. These transformations can be termed as follows:

1. Spacetime translations

   $\epsilon = a^\mu$.

2. Rotations

   $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$, ω antisymmetric.

3. Scale transformations

   $\epsilon^\mu = \lambda x^\mu$, $\lambda > 0$.

4. Special conformal transformations (SCT)

   $\epsilon^\mu = b^\mu x^2 - 2x^\mu(b \cdot x)$.

Using the below mentioned theorem we can check what finite or global transformations are generated by the above mentioned infinitesimal conformal transformation.

Theorem
Every conformal transformation that acts on an connected subset of Minkowski space, including the whole space itself, $\varphi : U \subset \mathbb{R}^{p,q}$, where $p + q > 2$, is a composition of

- a translation

  $x^\mu \to x^\mu + a^\mu$, where $a \in \mathbb{R}^d$,

- an orthogonal transformation (rotation)

  $x \to \Lambda x$, where $\Lambda \in O(p, q)$,

- a dilation (scale)

  $x^\mu \to \lambda x^\mu$, where $\lambda \in \mathbb{R}^+$,

- and an SCT

  $x \to \frac{x^\mu - bx^2}{1-2b\cdot x+b^2x^2}$, where $b \in \mathbb{R}^q$.

Note that SCT is non invertible and hence these transformations do not form a group. However if we compactify spacetime by adding ∞ as a point we can form a group containing global transformations. Therefore we can say that in some way SCT connects infinties to finite points in the spacetime.

2.2.2 Classification of Infinitesimal Conformal Translations for $d = 2$

For $d = 2$ the spacetime metric becomes the identity,

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{2.2.11}$$

And $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$ become the Cauchy-Riemann equations and $\epsilon(x)$ is complex-valued, complex-differentiable, and analytic. Therefore we introduce the complex coordinates,

$$z = x^1 + ix^2 \text{ and } \bar{z} = x^1 - ix^2. \tag{2.2.12}$$

Therefore we can write,

$$\epsilon(z) = \epsilon^1 + i\epsilon^2 \text{ and } \bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2. \tag{2.2.13}$$

Therefore as a first guess we can say that for $d = 2$ infinitesimal conformal transformations are some kind of infinitesimal analytic functions. The global conformal transformations thus correspond to entire holomorphic function $z \to f(z)$ with holomorphic inverses $f^{-1}(z)$.

We extend our manifold to a complex plane $\mathbb{C}$. Further to include infintiy as a point we compactify this complex space to a Riemann sphere.

$$\mathbb{R}^{2,0} \to \mathbb{C} \to \mathbb{C} \cup \{\infty\} \to \mathrm{Conf}(\mathbb{C} \cup \{\infty\}) \tag{2.2.14}$$

This compactification of spacetime to a Riemann sphere can be represented by the Figure 3.1.

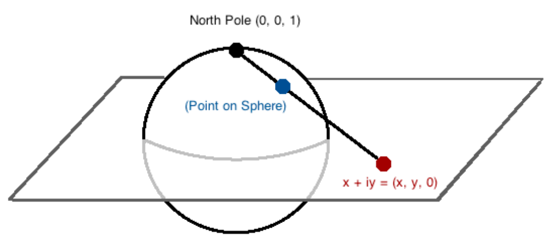

Figure 2.1: Riemann Sphere

The conformal group over this Riemann sphere is given by,

$$\mathrm{Conf}(\mathbb{C} \cup \{\infty\}) = \left\{ f(z) = \frac{\alpha z + \beta}{\gamma z + \delta};\ \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma \neq 0 \right\}. \tag{2.2.15}$$

This is also called a group of Mobius transformations and is a larger group than the general conformal group.

In this section we defined conformal groups as well as conformal transformations. We also defined four such transformations namely the translations, rotations, dilations and SCTs. We specifically looked at conformal groups in $d = 2$ and hence defined compactified spacetime. In the next section we extend this analysis of conformal transformations in $d = 2$ to form conformal algebra. We will also attempt to connect quantum mechanics with this conformal algebra.

2.3 Conformal Algebra in $d = 2$

In this section we specialise our discussion of conformal transformations to the case of $d = 2$. Also we will have a brief overlook of the constraints generated by the conformal symmetry with the help of a few concepts of quantum mechanics.

Let us consider a quantum system symmetric under a group G. To describe such a quantum system we need to find unitary transformations. For this we demand that G is a Lie Group. The algebra thus formed over such a group is termed as Lie algebra. We aim to relate Lie algebra with the conformal symmetry. Therefore for the quantum system having Hilbert space $\mathcal{H}$ which is symmetric under Lie group G following Lie algebra $\mathfrak{g}$ we need to find representations given by

$$(\mathcal{H},\ \pi : \mathfrak{g} \to L(\mathcal{H})) \tag{2.3.1}$$

here $L(\mathcal{H})$ is the set of linear operators, π generates a unitary operator on the Hilbert space such that $\pi(X) = e^{isX}$, $s \in \mathbb{R}$.

2.3.1 Local algebra of infinitesimal conformal transformations

In the previous section we showed that for global conformal transformations we have analytic functions over z given by $z \to f(z)$, where f is holomorphic with inverse f^{-1}. We now focus on infinitesimal transformations, such transformations can be given by,

$$z \to z + \epsilon(z) \text{ and } \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) \tag{2.3.2}$$

where ϵ is a holomorphic function. Hence ϵ will depend only on values of z and not on $\bar{z}$. For such transformations we can choose a convenient basis given by,

$$\epsilon_n(z) = -\epsilon z^{n+1}, \text{ where } n \in \mathbb{Z}. \tag{2.3.3}$$

For vector field corresponding to such infinitesimal transformations we must have a diffeomorphism given by,

$$z \to z + \epsilon(z) = exp(\epsilon X) \tag{2.3.4}$$

Here X is a vector field which is a tangent vector to every point in the manifold. Such a vector field can be described by differential operators. Thus we can write,

$$\ell_n \equiv -z^{n+1}\partial_z \text{ and } \bar{\ell}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{2.3.5}$$

These are differential operators corresponding to Lie algebra of infinitesimal transformations under infinite dimensional symmetries.
We can show that such operators do form a basis since they follow the below mentioned commutation relations,

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \tag{2.3.6}$$
$$[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n} \tag{2.3.7}$$
$$[\bar{\ell}_m, \ell_n] = 0. \tag{2.3.8}$$

The above mentioned commutation relations describe the Lie algebra of ℓ_m, ℓ_n closed under infinitesimal transformations. This is also known as Witt algebra. We can represent the Witt algebra by, Witt $= \mathcal{A} \oplus \bar{\mathcal{A}}$, where $\mathcal{A}$ is generated by $\{\ell_n\}$, and $\bar{\mathcal{A}}$ is generated by $\{\bar{\ell}_n\}$
Let us now try to generate $\{\ell_n\}$ corresponding to global transformations. Consider a vector field given by,

$$v(z) = -\sum_{n=-\infty}^{\infty} v_n \ell_n = \sum_{n=-\infty}^{\infty} v_n z^{n+1}\partial_z. \tag{2.3.9}$$

Such a vector field will exponentiate to form a holomorphic function f only if the vector field is non singular as $z \to 0$. This will in turn place a constraint on coefficients of vector field given by,

$$v_n = 0,\ n < -1. \tag{2.3.10}$$

Now the inverse of the vector field must also exponentiate to the holomorphic inverse f^{-1}. This happens only if the inverse is non singular in the limit $z \to \infty$. Hence the constraint arising on the vector field would be,

$$v_n = 0,\ n > 1. \tag{2.3.11}$$

Hence for Lie algebra under infinitesimal transformations to exponentiate to a holomorphic global transformation as well as holomorphic inverse global transformations only three conditions suffice. These three (six if we consider complex conjugates) generators of global conformal transformations are given by,

$$\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}. \tag{2.3.12}$$

These generators generate the group of linear fractional (Mobius) transformations, also known as the projective special linear group PSL$(2, \mathbb{C})$ are given by,

$$z \to \frac{az+b}{cz+d}, \ \textit{such that } ad - bc = 1. \tag{2.3.13}$$

The set of such global transformations can be expressed as,

$$\text{Translation:} \qquad e^{s\ell_{-1}} \equiv \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \qquad \equiv z \to z - s \tag{2.3.14}$$

$$\text{Dilation:} \qquad e^{s(\ell_0+\bar{\ell}_0)} \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \equiv z \to e^{-s} \tag{2.3.15}$$

$$\text{Rotation:} \qquad e^{is(\bar{\ell}_0-\ell_0)} \equiv \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \qquad \equiv z \to e^{is} \tag{2.3.16}$$

$$\text{Special Conformal:} \qquad e^{s\ell_1} \equiv \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \qquad \equiv z \to \frac{z}{1+cz}. \tag{2.3.17}$$

2.3.2 Minkowski spacetime ($d = 2$)

To apply whatever we have learnt so far about conformal transformations, let us consider the case of Minkowski spacetime $\mathbb{R}^{1,1}$. We can do so by describing the two theorems which are mentioned below.

Theorem:
A smooth map $\varphi = (u, v) : M \to \mathbb{R}^{1,1}$ from a connected subset of $M \subset \mathbb{R}^{1,1}$ is conformal , iff $u_x^2 > v_x^2$ and $u_x = v_y$, $u_y = v_x$ or $u_x = -v_y$, $u_y = -v_x$.

Theorem:
Consider an infinitely differentiable function on the real line $f \in C^\infty(\mathbb{R})$, and let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$, the infinitely differentiable functions from the real line to the real plane, be defined by $f_\pm(x, y) = f(x \pm y)$. Then the map

$$\Phi : C^{\infty}(\mathbb{R}) \times C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}^2, \mathbb{R}^2) \tag{2.3.18}$$

$$(f, g) \to \frac{1}{2}(f_+ + g_-, f_+ - g_-) \tag{2.3.19}$$

Has the following properties

- $\text{image}(\Phi) = \{(u, v) : \ u_x = v_y, \ u_y, v_x\}$
- $\Phi(f, g)$ is conformal iff $f' > 0$ and $g' > 0$ or $f' < 0$ and $g' < 0$
- Φ is bijective iff f and g are bijective
- $\Phi(f \circ h, g \circ k) = \Phi(f, g) \circ \Phi(h, k), \forall f, g, h, k \in C^{\infty}(\mathbb{R}) \equiv \Phi$ is a homomorphism.

The group of orientation-preserving transformations of $M = \mathbb{R}^{1,1}$ is isomorphic to

$$(\text{Diff}_+(\mathbb{R}) \times \text{Diff}_+(\mathbb{R})) \cup (\text{Diff}_-(\mathbb{R}) \times \text{Diff}_-(\mathbb{R})) \tag{2.3.20}$$

which consists of the infinitely-differentiable orientation-preserving maps of $\mathbb{R}$, diffeomorphisms of $\mathbb{R}$.

Such a symmetry when applied on Minkowski spacetime can be represented (roughly) by the below given Figure 3.2

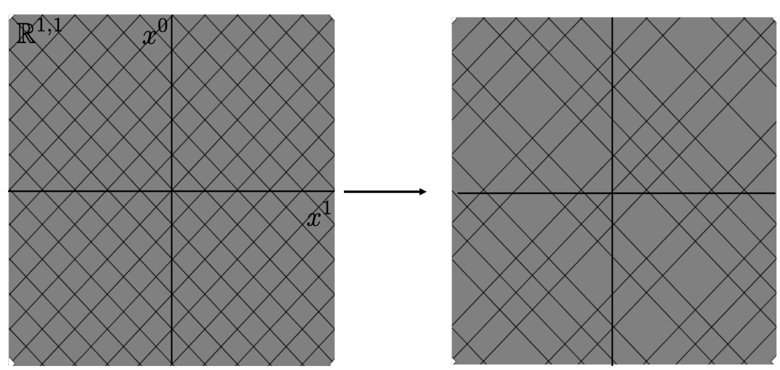

Figure 2.2: Representation of Minkowski spacetime under conformal symmetry

As mentioned in section 2.2 it is conveninent to compactify spacetime. If we compactify Minkowski spacetime the representation thus generated can be given by, $\mathbb{R}^{1,1} \to S^{1,1} \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2}$ Then the group of orientation-preserving transformations of $M = S^{1,1}$ is isomorphic to

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \equiv (\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)) \cup (\mathrm{Diff}_-(S^1) \times \mathrm{Diff}_-(S^1)). \quad (2.3.21)$$

Since we are at the very initial stages of building a conformal field theory, without focusing on mathematical intricacies we can examine briefly the implications of the above generated group for compactified Minkowski spacetime. Once we form the Lie algebra and try to exponentiate the locally conformal invariant quantum theories, it would generate a Hamiltonian without a lower bound. This is not permissible for any quantum theory. Hence instead of generating unitary transformations we will generate projective unitary transformations. All these projective transformations will be classified by a special (quantum) number known as central charge. By using central charge we can generate conformally invariant quantum theories. In this thesis we will gradually move towards these concepts.

In this section we extended the investigation of conformally invariant theories for the case d=2. We also tried to examine the relation of quantum theories and conformal transformations. Finally we took the example of Minkowski spacetime and had a brief overview of some new concepts which we will introduce gradually to reach the final aim of building a conformal field theory.

2.4 Classical Conformal Field Theories

In this section we will exhibit the constraints which will be placed on a classical field theory, if it is conformally invariant. These constraints will in turn help us in the coming sections of this thesis for building a quantum CFT.
Consider the conformal group $G = \mathrm{Conf}(\mathbb{R}^{p,q})$. In the last sections we identified four transformations belonging to G, namely translations, dilations, rotations and special conformal transformations. Over a manifold $\mathbb{R}^{p,q}$ these transformations can be extended as temporal translations, spatial translations, Lorrentz boosts, dilations and special conformal transformations. So for a conformal theory we have a built in hamiltonian H since it is a generator of time translations. Since G contains boosts, this compels us to consider a relativistically invariant conformal theory. This further constraints the theory to exhibit only certain symmetries.
Further we demand the theory to be local. This means we have a collection of observables $\phi_a(x)$, where $x \in \mathbb{R}^{p,q}$ and $a \in I$. These observables can be classical or quantum operators. We note here that the locality of the conformal field theory is the most important property which will help us to find irreducible representations of G. So, for the sake of this section let us call these local set of observables namely $\phi_a(x)$ as classical fields.

2.4.1 Classical Field Representations of Conformal Symmetry

The action for a classical field theory can be given by,

$$S = \int d^d x\, \mathcal{L}(\phi, \partial_\mu \phi), \text{ where } \phi = \{\phi_a(x)\}. \tag{2.4.1}$$

We can define a symmetry transformation for such an action as a mapping between the spacetime location and its image under the transformation $x \to x'$. We will assume the active field transformations i.e. the fields will aslo transform. Such a field transformation can be represented by,

$$\phi(x) \to \phi'(x') \equiv \mathcal{F}(\phi(x)) \tag{2.4.2}$$

Here $\mathcal{F}(\phi(x))$ implies that the transformed field depends on previous field configuration.

Under such a symmetry transformation, the action transforms as:

$$S' = \int d^d x \left|\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\right| \mathcal{L}\left(\mathcal{F}\left(\phi(x)\right), \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_{\nu}\mathcal{F}\left(\phi(x)\right)\right) \tag{2.4.3}$$

If we consider that the classical field theory, described under the symmetry transformation mentioned above, is conformally invariant then the Lagrangian density is allowed to change only by total derivative i.e.

$$\mathcal{L}' = \mathcal{L} + \text{total derivative.} \tag{2.4.4}$$

2.4.2 Infinitesimal Generators of Conf($\mathbb{R}^{p,q}$)

Now that we have defined the transformations of Lagrangian density and action under conformal symmetry, let us analyse the infinitesimal generators of conformal group. We define the following generators:

- Translation

  $P_{\mu} = -i\partial_{\mu}$

- Dilation

  $D = -ix^{\mu}\partial_{\mu}$

- Rotation (Boost)

  $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$

- Special Conformal

  $K_{\mu} = -i(2x_{\mu}x'^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

The commutation relations of the above defined generators will form a Lie algebra given by:

$$[D, P_{\mu}] = iP_{\mu} \tag{2.4.5}$$
$$[D, K_{\mu}] = -iK_{\mu} \tag{2.4.6}$$
$$[K_{\mu}, P_{\nu}] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \tag{2.4.7}$$
$$[K_{\rho}, L_{\mu\nu}] = i(\eta_{\rho\mu}K_{\nu} - \eta_{\rho\nu}K_{\mu}) \tag{2.4.8}$$
$$[P_{\rho}, L_{\mu\nu}] = i(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu}) \tag{2.4.9}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \tag{2.4.10}$$

Note: The other relations commute.

2.4.3 Field transformations under Conf($\mathbb{R}^{p,q}$)

Poincare group is a subgroup of conformal group. Therefore, a field transforming under Conf($\mathbb{R}^{p,q}$) always transforms under the subgroup:Poincare. Such a transformation is given by:

$$\phi_a(x) \to \sum_b \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x). \tag{2.4.11}$$

We focus on the subgroup of Conf($\mathbb{R}^{p,q}$) which leaves the origin fixed i.e. rotations, dilations and SCTs. The infinitesimal generators of this subgroup form a subalgebra by exponentiation,

$$\Lambda = e^{i\omega^\alpha G_\alpha}, \text{ where } \omega^\alpha \text{ is infinitesimal.} \tag{2.4.12}$$

The group element being $G_\alpha : K_\mu, D, L_{\mu\nu}$. At the origin the Poincare field transformation is given by:

$$\phi_a(x=0) \to \sum_b \pi(e^{i\omega^\alpha G_\alpha})_{ab}\phi_b(\Lambda^{-1}x=0) \tag{2.4.13}$$

Let us rename the group elements:

$$\pi(D) = \tilde{\Delta} \text{ (scaling dimension)} \tag{2.4.14}$$
$$\pi(K_\mu) = \kappa_\mu \tag{2.4.15}$$
$$\pi(L_{\mu\nu}) = S_{\mu\nu} \text{ (spin).} \tag{2.4.16}$$

The commutation relations of these generators are:

$$[\tilde{\Delta}, S_{\mu\nu}] = 0 \tag{2.4.17}$$
$$[\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \tag{2.4.18}$$
$$[\kappa_\mu, \kappa_\nu] = 0. \tag{2.4.19}$$

Now we will use this subalgebra to find the generators. Assume that $S_{\mu\nu}$ are irreducible representations then from Schurrs Lemma and above defined commutation relation (4.17) we can show that,

$$\tilde{\Delta} \propto \mathbb{I} \implies -i\kappa_\mu = 0. \tag{2.4.20}$$

We can use the above relation to show how dilations act on the fields. The coordinate transformation under dilation is given by,

$$x \to \lambda x \tag{2.4.21}$$
$$x \to \lambda^\epsilon \lambda^\epsilon \ldots \lambda^\epsilon x \tag{2.4.22}$$

For such a dilation the field transformation at origin is given by:

$$\phi_a(0) \to \lambda^{-\Delta_a}\phi_a(0) \tag{2.4.23}$$

For every conformal field transformation we can define the the Jacobian and scaling dimension as:

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda^{-\frac{d}{2}}, \text{ where } \Lambda = \lambda^{-2}. \tag{2.4.24}$$

Therefore the field transformation can be given by:

$$\phi'_a(x) = \lambda^{-\Delta}\phi_a(\lambda^{-1}x). \tag{2.4.25}$$

By using Baker-Campbell-Hausdorff formula we can show that,

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \tag{2.4.26}$$

$$\implies D\phi_a(x) = (-ix^\nu \partial_\nu + \tilde{\Delta})\phi_a(x). \tag{2.4.27}$$

In this section we attempted to form the Lie algebra of infinitesimal generators of the conformal group over a classical field theory. We saw that using this Lie algebra we were able to find the generators of conformal group. Now in the next section we will make the first attempt to study the action of conformal transformation over a quantum field theory.

2.5 Constraints of Conformal Invariance on Quantum Field Theories

We have defined the conformal group and the conformal transformations in the previous sections. Using these we also attempted to find the transformation of classical fields under conformal symmetry. Now we will make the attempt to find the transformation of quantum fields in this section. We will define some new type of field known as quasi-primary field and see what are the constraints placed over it by the conformal symmetry.

2.5.1 Assumptions for a QFT

Instead of actually quantising the field and forming a quantum field theory, we assume that the quantum field theory is a subset of all quantum theories. Since it is a quantum theory, we already have the following data:

- The theory has a kinematical space given by Hilbert space $\mathcal{H}$.
- We define a projective unitary representation of the conformal group $U(g)$, where $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$
- We define a vaccum vector $|0\rangle \in \mathcal{H}$ such that it is invariant upto a phase under global symmetries i.e. $U(g)|0\rangle = e^{i\varphi(g)}|0\rangle$.
- Using $|0\rangle$ we can generate the whole $\mathcal{H}$.
- For observables, we demand that we can measure some set of local observables from the set of self-adjoint linear operators on the Hilbert space

  $$A_{j,x} \in L_{\text{self-adjoint}}(\mathcal{H});\ x \in \mathbb{R}^{p,q};\ j \in \text{index set (e.g., particle type)}.$$

We will use these assumptions to study the constraints placed by the conformal transformations over a quantum field.

2.5.2 Quasi-Primary Fields

The *quasi-primary* of a field is a subset of local observables $\{A_{j,x} : j \in J, x \in \mathbb{R}^{p,q}\}$ with the additional properties to satisfy the assumed constraints that enforce conformal invariance of the system, denoted here by $\{\hat{\phi}_k(x) : k \in K\}$ that transform as

$$U(g): \ \hat{\phi}_k(x) \rightarrow U^\dagger(g)\hat{\phi}_k(x)U(g) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_k}{d}} \hat{\phi}_k(x') \tag{2.5.1}$$

Where $x' = gx$, and $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$ is a conformal transformation.

2.5.3 Correlation Functions

For the above defined quasi-primary fields the n-point correlation function under a conformal transformation is given by:

$$\langle 0| \hat{\phi}_{k_1}(x_1) \dots \hat{\phi}_{k_n}(x_n) |0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_{k_1}}{d}} \dots \left|\frac{\partial x_n'}{\partial x_n}\right|^{\frac{\Delta_{k_n}}{d}} \langle 0| \hat{\phi}_{k_1}(x_1') \dots \hat{\phi}_{k_n}(x_n') |0\rangle . \tag{2.5.2}$$

The above equation constraints the structure of n-point functions. Let us now try to study these constraints by analysing each type of conformal transformation.

Translations: For $x_j, x_k \in \mathbb{R}^{p,q}$, $j, k = 1, \dots, n$, the difference $x_j - x_k$ is invariant, and there are $d(n-1)$ such quantities.

Rotations: For spinless objects (in large enough dimension d), the length $r_{jk} \equiv |x_j - x_k|$ is invariant, and there are $\binom{n}{2}$ such quantities.

Dilations: Under the scale transformations, the length r_{jk} is clearly not invariant, but the ratio $\frac{r_{jk}}{r_{lm}}$ can be invariant.

SCTs: Under special conformal transformations, invariant quantities must be cross ratios of the form $\frac{r_{jk}r_{lm}}{r_{jl}r_{km}}$, since the squared length under SCTs transforms as

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)}. \tag{2.5.3}$$

Since all these are the invariant quantities, the correlation n-point function must be a function of these quantities to satisfy the respective covariance.

Two point correlation functions

Consider the two point function given by Green's function as shown below,

$$G^{(2)}(x_1,x_2) = \langle 0|\,\phi_1(x_1)\phi_2(x_2)\,|0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_{k_1}}{d}} \left|\frac{\partial x_2'}{\partial x_2}\right|^{\frac{\Delta_{k_2}}{d}} \langle 0|\,\hat{\phi}_{k_1}(x_1')\hat{\phi}_{k_2}(x_2')\,|0\rangle\,. \tag{2.5.4}$$

Under the action of translation, rotation, and dilation, we find that the Green's function is constrained to the form,

$$\langle 0|\,\phi_1(x_1)\phi_2(x_2)\,|0\rangle = \frac{c_{12}}{r_{12}^{2\Delta}}. \tag{2.5.5}$$

Where c_{12} is a constant determined by the normalization condition.
The SCT will put a further constraint that the scaling dimension must be same for the two fields. Therefore finally the two point correlation function becomes:

$$G^{(2)}(x_1,x_2) = \frac{f_{-\Delta_1-\Delta_2}}{r_{12}^{\Delta_1+\Delta_2}} = \frac{c_{12}}{r_{12}^{\Delta_1+\Delta_2}} \tag{2.5.6}$$

So, we find that for two quasi primary fields we can say that they are conformally invariant only if they have same scaling dimension.

Three-point correlation function

Proceeding in the similar manner as for the above example we can give the three point correlation function under translations, rotations and dilations by

$$\langle 0|\,\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\,|0\rangle = \sum_{a,b,c} \frac{c_{abc}}{r_{12}^a r_{23}^b r_{31}^c} \tag{2.5.7}$$

Where the summation is constrained by the field scaling dimensions to satisfy $a+b+c = \Delta_1+\Delta_2+\Delta_3$.
Special conformal transformations further constrain the exponents in the power law to

$$a = \Delta_1+\Delta_2-\Delta_3, \quad b = -\Delta_1+\Delta_2+\Delta_3, \quad c = \Delta_1-\Delta_2+\Delta_3. \tag{2.5.8}$$

And the three-point correlation function has the form

$$\langle 0|\,\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\,|0\rangle = \frac{c_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{-\Delta_1+\Delta_2+\Delta_3} r_{31}^{\Delta_1-\Delta_2+\Delta_3}}. \tag{2.5.9}$$

Four-point correlation functions

The four-point correlation function has the form

$$\langle 0|\,\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\,|0\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{13}r_{24}}{r_{23}r_{14}}\right)\prod_{j<k} r_{jk}^{-\Delta_j-\Delta_k+\frac{\Delta}{3}} \tag{2.5.10}$$

Where $F(\cdot,\cdot)$ is an arbitrary function of the cross products and $\Delta = \sum_j \Delta_j$.

2.5.4 Correlation functions for d=2 case

Let us consider the example of a quantum field theory in (2+0)d dimensions. For this case we had defined the imaginary spacetime coordinates as well as the fields depending on those coordinates in Section 2.3. These transformations were given by,
For $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$

$$\Phi(x_1, x_2) = \Phi(z, \bar{z}) \tag{2.5.11}$$

By analogy we can extend the definition of quasi primary field. Let us refer to them as primary fields of type $(h, \bar{h})$. Under conformal transformation such fields will transform as,

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{2.5.12}$$

Two point correlation function

We define the two point correlation function as,

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \langle 0|\,\Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\,|0\rangle \tag{2.5.13}$$

Where $\underline{z} \equiv (z_1, z_2)$ and $\underline{\bar{z}} \equiv (\bar{z}_1, \bar{z}_2)$.

From the above equation we can consider each type of transformation and thus obtain the constraints which are invariant.
Translation:

$$\epsilon(z) = \epsilon \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) \propto z_{12} = z_1 - z_2 \text{ and } \bar{z}_{12} = \bar{z}_1 - \bar{z}_2. \tag{2.5.14}$$

Rotation & Dilation:

$$\epsilon(z) = \epsilon z \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{h_1+h_2}\ \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}. \tag{2.5.15}$$

SCT:

$$\epsilon(z) = z^2 \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{2h}\ \bar{z}_{12}^{2\bar{h}}}. \tag{2.5.16}$$

For example consider the bosonic field in (2+0)d. Such a field has $h = \bar{h}$. Setting the condition $\Delta = h + \bar{h}$ the two point correlation function for a bosonic field becomes,

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{|z_{12}|^{2\Delta}}. \tag{2.5.17}$$

Three point correlation function

We can similarly derive the three point correlation function for quasi primary field in (2+0)d.

$$G^{(3)}(\underline{z}, \underline{\bar{z}}) = c_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{31}^{h_1-h_2+h_3}} \frac{1}{\bar{z}_{12}^{h_1+h_2-h_3} \bar{z}_{23}^{-h_1+h_2+h_3} \bar{z}_{31}^{h_1-h_2+h_3}} \tag{2.5.18}$$

In this section we considered the conformal invariance of quantum field theory particularly the quasi primary fields. We derived the correlation functions and analysed the constraints placed by those functions. So, we have made a whole set of rules to conclude whether any QFT is conformally invariant. We derived these rules in the form of correlation functions. We will use these rules to understand the physical implications on the quantum theory which is conformally invariant in the upcoming sections.

2.6 Quantum CFTs: Radial Quantisation and Ward identities

In the last few sections we have developed the constraints for classical as well as quantum field theories under the action of conformal symmetry. We particularly focused on the case of d=2 case i.e. the case of Euclidean manifold. However we now aim to find the physical implications of conformal symmetry for a quantum theory in Minkowski spacetime. We hope to carry over the same concepts which we defined in an Euclidean manifold to define a quantum CFT in Minkowski spacetime.

2.6.1 Analytic Continuation

In previous sections we found that conformal field theory yields (projective) unitary representations of Conf($\mathbb{R}^{2,0}$). We wish to study such theories over a Minkowski manfold.The (projective) unitary representations of Conf($\mathbb{R}^{2,0}$) were defined on a complex plane (Riemann sphere). Therefore, we can use analytic continuation to map such theories to a Minkowski theory where we would have representations of Conf($\mathbb{R}^{1,1}$). Such analytic continuation of coordinates can be given by,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ it \end{pmatrix};\ t \in \mathbb{R}. \tag{2.6.1}$$

Now suppose that the correlation functions are analytically continued in the same manner:

$$\langle 0|\, \hat{\phi}(x,y)\hat{\phi}(0,0)\, |0\rangle = f(x,y) \to f(x,it). \tag{2.6.2}$$

Now if we analytically continue all n-point correlation functions in the above manner it often happens that a conformally invariant QFT in $\mathbb{R}^{2,0}$ gets transformed to a conformally invariant QFT in $\mathbb{R}^{1,1}$. We therefore get a mapping given by,

$$G^{(n)}((x_1,y_1),\dots) \to G^{(n)}((x_1,t_1),\dots) \tag{2.6.3}$$

The final aim is to study the physical implications of such a conformally invariant QFT in $\mathbb{R}^{1,1}$. We have studied the properties of conformal field theory in Euclidean manifold.

Now we need to study the same for a Minkowski-CFT. We approach this problem by reversing the analytic continuation. We will now analytically continue the representations of Conf $\mathbb{R}^{1,1}$ to representations of Conf $\mathbb{R}^{2,0}$. The correlation function for a Minkowski-CFT is given by

$$\langle 0| \hat{\phi}(x,t)\hat{\phi}(0,0) |0\rangle \tag{2.6.4}$$

Considering such a correlation function, enter Heisenberg picture to apply time evolution, and analytically continue the time variable to define inverse temperature $t \to -i\beta$.

$$\langle 0| \hat{\phi}(x,y)\hat{\phi}(0,0) |0\rangle = \langle 0| e^{-it\hat{H}}\hat{\phi}(x,0)e^{it\hat{H}}\hat{\phi}(0,0) |0\rangle \tag{2.6.5}$$

$$= \langle 0| e^{-\beta\hat{H}}\hat{\phi}(x,0)e^{\beta\hat{H}}\hat{\phi}(0,0) |0\rangle . \tag{2.6.6}$$

The resulting correlation function represents the thermal state of the Hamiltonian H with inverse temperature β.

2.6.2 Radial Quantisation

For a Minkowski CFT let us label the spacetime coordinates as $(\sigma^0, \sigma^1) \in \mathbb{R}^{1,1}$. We can complexify these spacetime coordinates by analytic continuation such that timelike coordinate is sent to imaginary corrdinate i.e. $\sigma^0 \to i\sigma^0$ and

$$z = \sigma^1 + i\sigma^0 \text{ and } \bar{z} = \sigma^1 - i\sigma^0 \tag{2.6.7}$$

Hence the analytically continued n-point correlation function becomes,

$$G^{(n)}(\underline{\sigma}_1, \underline{\sigma}_2, \dots) \to G^{(n)}(\underline{z}_1, \underline{\bar{z}}_1; \underline{z}_2, \underline{\bar{z}}_2; \dots). \tag{2.6.8}$$

The complexified Euclidean spacetime generated above has coordinates given by z and $\bar{z}$. We can compactify spacetime to that of a cylinder such that space corresponds to the transverse direction $\sigma^1 \to \sigma^1 + 2\pi$, and time corresponds to the longitudinal direction. We thus get a cylinder.
Next we map the cylinder to a complex plane by writing the coordinates as,

$$z = e^{\sigma^1 + i\sigma^0} \text{ and } \bar{z} = e^{\sigma^1 - i\sigma^0}. \tag{2.6.9}$$

The infinite past and future on the cylinder are mapped on the complex plane as,

$$\sigma^0 = -\infty \to z = 0 \text{ and } \sigma^0 = \infty \to z = \infty \tag{2.6.10}$$

Such a compactification is shown in Figure 2.3. Circles on the complex plane correspond to constant time.
The transformation maps can be given by:

$$\text{Time reversal } [\sigma^0 \to -\sigma^0] \Rightarrow \text{Inversion } [z \to \frac{1}{z}] \tag{2.6.11}$$

$$\text{Time translation } [\sigma^0 \to \sigma^0 + a] \Rightarrow \text{Dilation } [z \to e^a z] \tag{2.6.12}$$

$$\text{Spatial translation } [\sigma^1 \to \sigma^1 + a] \Rightarrow \text{Rotation } [z \to e^{ia} z] \tag{2.6.13}$$

The dilation generator on conformal plane can be regarded as the Hamiltonian for the system and Hilbert space is developed on constant radius surfaces. The whole process of defining the quantum CFT on this complex plane is referred to as radial quantisation.

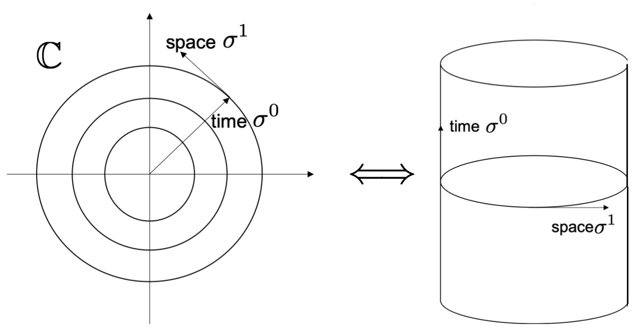

Figure 2.3: Compacification of spacetime

In this section we made an attempt to form a Quantum CFT in Minkowski spacetime. We used analytical continuation to relate Euclidean CFTs and Minkowski CFTs. Finally we developed a method to study CFT in Minkowski spacetime known as radial quantisation. We now move forward to develop a CFT in complex plane.

2.7 Conformal Ward identities

We wish to study the generators of conformal symmetry in Minkowski space-time. For this we will define Ward identities in classical as well as quantum CFTs. These identities will in turn help us to study the generators of conformal transformations. We begin by the classical case.

2.7.1 Ward identities for classical field theory

We will employ path integral heuristic approach to quantise a classical conformal field theory and hence find the quantum generators of conformal symmetry. Suppose we have a classical field theory with action $S[\underline{\phi}]$, where $\underline{\phi}$ is a vector of classical fields. Assume that S is symmetric under infinitesimal transformations, such that

$$\underline{\phi}'(\underline{x}) = \underline{\phi}(\underline{x}) - i\omega_a(\underline{x})\mathbf{G}_a\underline{\phi}(\underline{x}) = e^{-i\omega_a(\underline{x})\mathbf{G}_a}\underline{\phi}(\underline{x}) \tag{2.7.1}$$

Where $\omega_a(\underline{x})$ is infinitesimal and $\mathbf{G}_a$ is a matrix acting on the vector labels of $\underline{\phi}$.
We use the path integral prescription to construct correlation functions and hence get a quantised version of the above define of classical fields,

$$\langle 0| \hat{\underline{\phi}}(\underline{x}_1)\dots\hat{\underline{\phi}}(\underline{x}_n)|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(\underline{x}_1)\dots\phi(\underline{x}_n)e^{iS[\underline{\phi}]}}{\int \mathcal{D}\phi\, e^{iS[\underline{\phi}]}}. \tag{2.7.2}$$

Under a conformal transformation we assume that $\underline{\phi} \to \underline{\phi}'$ and integral measure being invariant $\mathcal{D}\phi' = \mathcal{D}\phi$. Also we define the operator $\hat{X}$ to be the time-ordered product of quantum fields,

$$\hat{X} = \mathcal{T}[\hat{\underline{\phi}}(\underline{x}_1)\dots\hat{\underline{\phi}}(\underline{x}_n)]. \tag{2.7.3}$$

The expectation value in the the expectation value of $\hat{X}$ in the path integral prescription is given by,

$$\langle\hat{X}\rangle = \frac{\int \mathcal{D}\phi' \left(\hat{\underline{\phi}}(\underline{x}_1)\dots\hat{\underline{\phi}}(\underline{x}_n) + i\omega_a(\underline{x})\mathbf{G}_a(\hat{\underline{\phi}}(\underline{x}_1)\dots\hat{\underline{\phi}}(\underline{x}_n) + \dots\right) e^{i(S+\int dx\, \partial_\mu j_a^\mu \omega_a(\underline{x}))}}{\int \mathcal{D}\phi\, e^{iS}}. \tag{2.7.4}$$

To first order in ω_a we get the identity given by,

$$\frac{\partial}{\partial x^\mu}\langle \hat{j}_a^\mu(\underline{x})\hat{\phi}(\underline{x}_1)\dots\hat{\phi}(\underline{x}_n)\rangle = -i\sum_{j=1}^{n}\delta(\underline{x}-\underline{x}_j)\langle\hat{\phi}(\underline{x}_1)\dots\mathbf{G}_a\hat{\phi}\left(\underline{x}_j)\dots\hat{\phi}(\underline{x}_n)\right)\rangle. \tag{2.7.5}$$

This is known as a Ward identity. It gives us the generator of symmetry as we proceed using path integral approach.
Let us now switch over to the quantum case and try to find the precise Ward identities for each of the four generators of conformal symmetry.

2.7.2 Conformal Ward Identities

We will proceed in the same way as in the previous subsection. However we will now work with "imaginary-time" $t = -i\beta$. Such a choice of analytic continuation is often termed as Wick rotation. Wick rotation maps the unitary generators of time translations to a non-unitary semigroup with fixed points being the ground state. Such a Wick rotation map can be given by,

$$U(t) = e^{-it\hat{H}} \to S(\beta) = e^{-\beta\hat{H}}. \tag{2.7.6}$$

Now suppose we have a tupple of classical fields $\phi(x)$. Let S be the action of the fields which we assume to be invariant under infinitesimal conformal transformations. Such infinitesimal conformal field transformations can be written as,

$$\phi(x) \to \phi'(x) = \phi(x) - i\omega_a(x)G_a\phi(x) \tag{2.7.7}$$

where G_a is a symmetry transformation matrix and hence represents the generators of conformal symmetry.
Note that time ordering is implicit when LHS is a correlator and RHS is a path integral. Following this we define:

$$\hat{X} \equiv \mathcal{T}[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)]. \tag{2.7.8}$$

Under a Wick rotation we can use the fact that,

$$\int \mathcal{D}\phi\, e^{iS} \to \mathrm{tr}(e^{-\beta H}) = Z. \tag{2.7.9}$$

here Z is the partition function. Using path integral formulation we can thus write,

$$\langle\hat{X}\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, X e^{iS[\phi]} = \frac{1}{Z}\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n)e^{iS[\phi]} \tag{2.7.10}$$

Now if we carry over the same steps as in the previous subsection we will get the following equation,

$$\frac{\partial}{\partial x^\mu}\langle \hat{j}^\mu_a(x)\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\rangle = -i\sum_{j=1}^{n}\delta(x-x_j)\langle\hat{\phi}(x_1)\ldots G_a\hat{\phi}(x_j)\ldots\hat{\phi}(x_n)\rangle \tag{2.7.11}$$

This is the desired expression for Ward identity. Ward identity is the principal tool for obtaining the quantised symmetries and showing how the symmetries are implemented on the correlation functions. We can interpret the conserved charge for such symmeties by the expression,

$$\hat{Q}_a = \int d^{d_a}x\ \hat{j}^0_a(x). \tag{2.7.12}$$

Here $\hat{j}_a(x)$ is the conserved current serving as the quantum generator of the symmetry.
Integrate the Ward identity now over a thin slice of time $t_- < t < t_+$ as shown in the Figure 3.3 given below. Also suppose that there is one point of time $x^0_1 = t$ that is contained within the slice, and all others are sufficiently far away.

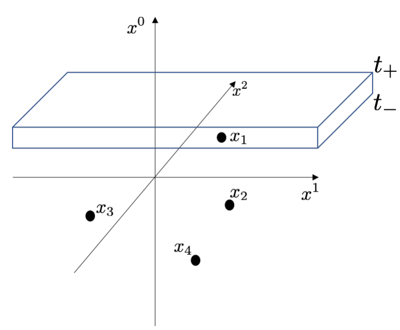

Figure 2.4: Time Slice to integrate the Ward identity in the box

Hence the LHS of the Ward identity would become,

$$\langle\hat{Q}_a(t_+)\hat{\phi}(x_1)\hat{Y}\rangle - \langle\hat{Q}_a(t_-)\hat{\phi}(x_1)\hat{Y}\rangle = -i\langle G_a\hat{\phi}(x_1)\hat{Y}\rangle \tag{2.7.13}$$

where $\hat{Y} = \hat{\phi}(x_2)\ldots\hat{\phi}(x_n)$.

Now if we limit the inverse temperature tending to infinity, we have $\langle \hat{X} \rangle \to \langle 0| \hat{X} |0\rangle$, we can show from equation (2.7.13) that the expression for Ward identity becomes:

$$\langle 0| [\hat{Q}_a, \hat{\phi}(x_1)]\hat{Y} |0\rangle = -i \langle 0| G_a\hat{\phi}(x_1) |0\rangle . \tag{2.7.14}$$

From the above equation we can show that,

$$[\hat{Q}_a, \hat{\phi}] = -iG_a\hat{\phi}. \tag{2.7.15}$$

We can insert the conserved charge for the concerning symmetry and find the generator of the transformation.

2.7.3 Quantum Generators for Conformal Symmetries

Using the expressions derived in the previous subsection we can find the Ward identities for each of the conformal symmetries. Let $\hat{X}$ be the product of local quantum field operators as before. The conserved current is the *energy-momentum tensor* $\hat{T}^{\mu}_{\nu}$, where we assume that T^{ν}_{μ} (classical) is traceless $T^{\mu}_{\mu} = 0$ and symmetric $T_{\mu\nu} = T_{\nu\mu}$.

Translation:
The Ward identity for the conformal transformation corresponding to translation can be given by:

$$\partial_\mu \langle \hat{T}^{\mu}_{\nu} \hat{X} \rangle = -i \sum_j \delta(x - x_j)\partial_{x^\nu_j} \langle \hat{X} \rangle. \tag{2.7.16}$$

Rotation:
For the case of rotation the conserved current is given by, $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$. Thus the Ward identity for the conformal transformation corresponding to rotations can be given by:

$$\langle (\hat{T}^{\rho\nu} - \hat{T}^{\nu\rho})\hat{X} \rangle = -i \sum_j \delta(x - x_j)s^{\nu\rho}_j \langle \hat{X} \rangle \tag{2.7.17}$$

Note that $s^{\nu\rho}_j$ is the spin matrix representation of the j^{th} field vector.

Dilation:
For the case of dilation, the generator of conformal symmetry can be given

by, $G \equiv D = -ix^{\nu}\partial_{\nu} - i\Delta$. Thus the Ward identity for the conformal transformation corresponding to dilations can be given by:

$$\langle \hat{T}^{\mu}_{\mu} \hat{X} \rangle = -\sum_{j} \delta(x - x_j)\Delta_j \langle \hat{X} \rangle. \tag{2.7.18}$$

Note that Δ_j corresponds to the scaling dimension for j^{th} field vector.

In this section we started with a classical field theory and quantised it using path integral formulation. Using the path integral formulation we could derive an expression which can hand us over the generator of the concerning symmetry, termed as Ward identitiy. Using the generators of conformal symmetry we finally derived the conformal Ward identities.

2.8 Conformal Ward identities in (2+0)d

We attempt to transform the Ward identities derived in the previous section to the special case of (2+0)d dimensions. In Section 2.5 we used Radial Quantisation and further compactified the cylinder to a complex plane in the case of analytic continuation from Minkowski CFT to a Euclidean CFT. Hence using the same ideas we specialise to (2+0)d dimensions using complex coordinates.

2.8.1 Complexifying coordinates

For specialising the conformal Ward identities to a complex plane we need to transform the quantities appearing in those identities using complex coordinates.
The complex coordinates are represented by $z = x + iy$ and $\bar{z} = x - iy$.
Using these complex coordinates the line element is given by,

$$ds^2 = g_{zz}dzdz + g_{z\bar{z}}dzd\bar{z} + g_{\bar{z}z}d\bar{z}dz + g_{\bar{z}\bar{z}}d\bar{z}d\bar{z} \tag{2.8.1}$$

$$= 0 \cdot dzdz + \frac{1}{2} \cdot dzd\bar{z} + \frac{1}{2} \cdot d\bar{z}dz + 0 \cdot d\bar{z}d\bar{z} = dzd\bar{z}. \tag{2.8.2}$$

Vector quantities transform after complexifying as:

$$F = F^x\partial_x + F^y\partial_y = F^z\partial_z + F^{\bar{z}}\partial_{\bar{z}} \tag{2.8.3}$$

where $F^z = F^x + iF^y$ and $F^{\bar{z}} = F^x - iF^y$.
Tensor quantities transform after complexifying as:

$$T_{zz} = \frac{1}{4}(T_{xx} - 2iT_{xy} - T_{yy}) \tag{2.8.4}$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{xx} + 2iT_{xy} - T_{yy}) \tag{2.8.5}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{xx} + T_{yy}). \tag{2.8.6}$$

Note that the stress energy tensor will be treated as symmetric.

2.8.2 Conformal Ward identities in complex plane

Using the above transformed quantities the Ward identities in complex coordinates become,

Translation, holomorphic: (2.8.7)

$$2\pi\partial_z\langle\hat{T}_{\bar{z}z}\hat{X}\rangle + 2\pi\partial_{\bar{z}}\langle\hat{T}_{zz}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)\partial_{w_j}\langle\hat{X}\rangle \tag{2.8.8}$$

Translation, anti-holomorphic: (2.8.9)

$$2\pi\partial_z\langle\hat{T}_{\bar{z}\bar{z}}\hat{X}\rangle + 2\pi\partial_{\bar{z}}\langle\hat{T}_{z\bar{z}}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{z}\left(\frac{1}{\bar{z}-\bar{w}_j}\right)\partial_{\bar{w}_j}\langle\hat{X}\rangle \tag{2.8.10}$$

Rotation: (2.8.11)

$$-2\pi\langle\hat{T}_{z\bar{z}}\hat{X}\rangle + 2\pi\langle\hat{T}_{\bar{z}z}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)s_j\langle\hat{X}\rangle \tag{2.8.12}$$

Dilation: (2.8.13)

$$2\pi\langle\hat{T}_{z\bar{z}}\hat{X}\rangle + 2\pi\langle\hat{T}_{\bar{z}z}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)\Delta_j\langle\hat{X}\rangle \tag{2.8.14}$$

Add and subtract the Ward identities for **rotation** and **dilation** and substitute them into the **holomorphic** and **antiholomorphic translation** Ward identities to get a more compact form. Finally we would get two identities depending either on z or $\bar{z}$. The one depending on the conjugate is termed as anti-holomorphic while the other is termed as holomorphic.

Holomorphic: (2.8.15)

$$\partial_{\bar{z}}\left(\langle\hat{T}(z,\bar{z})\hat{X}\rangle - \sum_j\left(\frac{1}{z-w_j}\partial_{w_j}\langle\hat{X}\rangle + \frac{h_j}{(z-w_j)^2}\langle\hat{X}\rangle\right)\right) = 0 \tag{2.8.16}$$

Anti-holomorphic: (2.8.17)

$$\partial_{z}\left(\langle\hat{\bar{T}}(z,\bar{z})\hat{X}\rangle - \sum_j\left(\frac{1}{\bar{z}-\bar{w}_j}\partial_{\bar{w}_j}\langle\hat{X}\rangle + \frac{h_j}{(\bar{z}-\bar{w}_j)^2}\langle\hat{X}\rangle\right)\right) = 0 \tag{2.8.18}$$

(2.8.19)

Here we have represented $\hat{T}(z,\bar{z}) = -2\pi\hat{T}_{zz}$ and $\hat{\bar{T}}(z,\bar{z}) = -2\pi\hat{T}_{\bar{z}\bar{z}}$. Also we have used the definitions of scaling dimension $\Delta_j = h_j + \bar{h}_j$ and the spin

$s_j = h_j - \bar{h}_j$.
Now we can cut off the dependence of $\bar{z}$ in $\hat{T}(z, \bar{z})$ by assuming that the dependence on $\bar{z}$ is due to some regular functions of z (since it is holomorphic identity). Similarly we can cut off the dependence of z from the expression for antiholomorphic identity. Once we do this we can write the above two identities as:

$$\langle \hat{T}(z)\hat{X}\rangle = \sum_j \left(\frac{1}{z - w_j}\partial_{w_j}\langle \hat{X}\rangle + \frac{h_j}{(z - w_j)^2}\langle \hat{X}\rangle \right) + (\text{regular functions of } z) \tag{2.8.20}$$

$$\langle \hat{\bar{T}}(\bar{z})\hat{X}\rangle = \sum_j \left(\frac{1}{\bar{z} - \bar{w}_j}\partial_{\bar{w}_j}\langle \hat{X}\rangle + \frac{h_j}{(\bar{z} - \bar{w}_j)^2}\langle \hat{X}\rangle \right) + (\text{regular functions of } \bar{z}) \tag{2.8.21}$$

Extracting particular Ward identities

We now have to use these identities to extract an expression such that it will give us particular identities for each conformal transformation.
Under an infinitesimal conformal transformation $x \to x + \epsilon$, we derived the following identity in Section 2.2
$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$
Using this we can derive,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \partial_\rho \epsilon^\rho \eta_{\mu\nu} \tag{2.8.22}$$

$$\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \epsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta. \tag{2.8.23}$$

Integrating $\partial_\mu(\epsilon_\nu T^{\mu\nu})$ and using Gauss' theorem we can express the integrated form of Ward identity as :

$$\delta_{\epsilon,\bar{\epsilon}} = \frac{1}{2} i \oint_C \left(-dz \langle \hat{T}^{\bar{z}\bar{z}} \epsilon_{\bar{z}} \hat{X}\rangle + d\bar{z}\langle \hat{T}^{zz}\epsilon_z \hat{X}\rangle \right). \tag{2.8.24}$$

In the above equation we can substitute for ϵ and extract Ward identity for each conformal transformation.

The equations derived in these section, particularly the compact form of Ward identities will be used to generate Lie Algebra of CFT. Also we will now specialise to the case of (2+0)d dimensions in the forthcoming sections, since we could derive two very compact forms of Ward identities and hence the algebra would be easy if we deal with the same case in further sections.

2.9 Radial Quaantisation and OPE

We now continue to discuss the process of radial quantisation to build a conformal field theory. The main objective is to develop the Lie algebra for such a CFT. We will introduce expressions known as Operator Product Expansion (OPE) which will be used to develop the Lie algebra of CFT.

2.9.1 Radial ordering

For building CFTs we employ the use of path integral formulation for quantising classical field theories. Such a path integral can be given by using time ordered correlator of fields:

$$\langle \mathcal{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]\rangle = \lim_{\beta\to\infty} \frac{\int \mathcal{D}\phi\, \phi(x_1)\ldots\phi(x_n)e^{-S[\phi]}}{\int \mathcal{D}\phi\, e^{-S[\phi]}}. \tag{2.9.1}$$

The time ordering here plays an important role while choosing coordinate system. Every choice of what coordinate is "time" leads to a different time ordering symbol. This in turn leads to a different Ward identity and ultimately a different realisation of Hilbert space.

While doing the analytic continuation of a Minkowski-CFT to an Euclidean-CFT the coordinate system can be realised as that on a cylinder. We further compactified such a cylinder to a complex plane in Section 2.6. On such a complex plane if we take the radial coordinate as β, the inverse temperature, then we deal with thermodynamics of a quantum system on a circle.

For defining such theories on a circle over a complex plane we can use polar coordinates such that the radial coordinate refers to the time coordinate. Considering polar coordinates if we solve path integrals we will get time ordered correlation function which on a circle would correspond to correlation functions with fields ordered in increasing radius. This is known as radial ordering. We define the radial ordering symbol as:

$$\mathcal{R}[\hat{A}(z)\hat{B}(w)] = \begin{pmatrix} \hat{A}(z)\hat{B}(w), & |z| > |w| \\ \hat{B}(w)\hat{A}(z), & |z| < |w| \end{pmatrix}. \tag{2.9.2}$$

Note that whenever we deal with product of field operators within a correlator, we have vacuum expectation values with radial ordering of field operators. For example:

$$\langle \hat{T}(z)\hat{A}(w)\rangle \equiv \langle 0|\, \mathcal{R}[\hat{T}(z)\hat{A}(w)]\, |0\rangle\,. \tag{2.9.3}$$

2.9.2 Commutators of Field operators

In radial coordinate system integrating equal time slices correspond to an integral over a contour. Using such contour integrals we can find the conserved charges and hence the field operators. Thus for finding commutators of these conserved charges we can use contour integrals.
The contour integral therefore will be over product of operators $\hat{a}(z)\hat{b}(w)\ldots$.
Let us build a conserved charge $\hat{A} = \oint dz\ \hat{a}(z)$. We aim to find the commutator $[\hat{A}, \hat{b}(w)]$. We can form commutators by first integrating over a contour with radius infinitesimally smaller than w and then subtracting it with contour integral infinitesimally larger than w. In Figure 3.4 given below , consider the following set of contours C_1 and C_2 in the complex plane surrounding the point $w = x + iy$ separated by a distance ϵ, the radius of the enclosed contour being w.

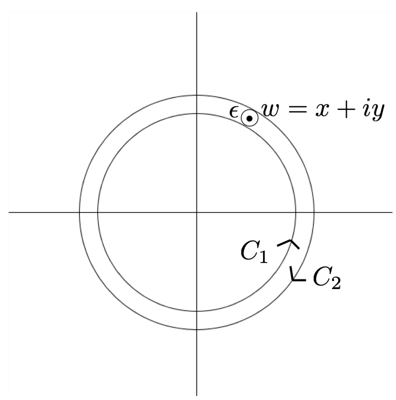

Figure 2.5: Contour integrals to obtain commutator

Therefore the commutator can be written as,

$$[\hat{A}, \hat{b}(w)] = \oint_{C_1} dz\ \hat{a}(z)\hat{b}(w) - \oint_{C_2} dz\ \hat{b}(w)\hat{a}(z). \tag{2.9.4}$$

$$= \oint_{W} dz\ \hat{a}(z)\hat{b}(w) \tag{2.9.5}$$

2.9.3 Product of primary field and stress-energy tensor

Let let $\hat{\phi}(w, \bar{w})$ be a primary field. Under an infinitesimal conformal transformation the field transforms as,

$$\delta_{\epsilon,\bar{\epsilon}}\hat{\phi}(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint dz\, \epsilon(z)\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] + \oint d\bar{z}\, \bar{\epsilon}(\bar{z})\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})]\right) \tag{2.9.6}$$

Employing the use of contour integral as shown in previous subsection 2.9.2, we can get

$$\delta_{\epsilon,\bar{\epsilon}}\hat{\phi}(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint_{C_1} dz\, \epsilon(z)\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] - \oint_{C_2} d\bar{z}\, \bar{\epsilon}(\bar{z})\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})]\right) \tag{2.9.7}$$

$$= \frac{1}{2\pi i}\left(\oint dz\, \epsilon(z)[\hat{T}(z), \hat{\phi}(w,\bar{w})] + \oint d\bar{z}\, \bar{\epsilon}(\bar{z})[\hat{\bar{T}}(\bar{z}), \hat{\phi}(w,\bar{w})]\right). \tag{2.9.8}$$

We claim that the above obtained expression is just another form of Ward identity. We can show that the above expression holds true only if we take the precise radial ordering as shown below:

$$\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] = \frac{h}{(z-w)^2}\hat{\phi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\phi}(w,\bar{w}) + \text{holomorphic functions} \tag{2.9.9}$$

$$\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})] = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\hat{\phi}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\hat{\phi}(w,\bar{w}) + \text{anti-holomorphic functions.} \tag{2.9.10}$$

Short distance approximation

As we bring the two fields closer at $z \to w$ the additional functions appearing in the RHS of holomorphic equation (2.9.9) tend to just some constant numbers and therefore can be neglected. Therefore at short distance approximation we can write,

$$\hat{T}(z)\hat{\phi}(w,\bar{w}) \sim \frac{h}{(z-w)^2}\hat{\phi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\phi}(w,\bar{w}). \tag{2.9.11}$$

Therefore, we have actually found the expansion of products of field operators. In particular the above expression has a product of two different fields which is equal to a sum of terms, each a single field operator,multiplied by possible singular complex functions. Such expressions are known as Operator Product Expansion(OPE).

2.9.4 Operator Product Expansion(OPE)

If we have two fields $\hat{A}(z)$ and $\hat{B}(w)$ at short distance approximation such that $z \to w$ then the OPE in general can be given by,

$$\hat{A}(z)\hat{B}(w) \sim \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{2.9.12}$$

where $\{AB\}_n(w)$ are composite fields non-singular at w.

In this section we used the process of radial quantisation to form commutators of field operators. These commutators are solved using contour integrals. Such commutators can be used to develop a theory for two short distance approximated fields which in turn leads to forming an OPE. The OPEs will be used in upcoming section to build the Lie Algebra of CFT.

2.10 The case of Mass-less Free Boson

Though we have not developed the conformal field theory completely, we can apply whatever we have learnt in the previous sections to the case of a mass-less free boson. We will form the OPE of the free bosonic field and check whether such a field follows the OPE of a primary field i.e. equation (2.9.11). If the OPE does match with equation (2.9.11) we will claim that the field is conformal, however if it doesn't we will try to generate the conformal field.

2.10.1 Dynamics of Free Bosonic field

We take the Lagrangian as that for the Klein-Gordon field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \tag{2.10.1}$$

For the case of mass-less free Boson we can write,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \tag{2.10.2}$$

We now specialise to Euclidean picture in (2+0)d and write the action in complex coordinates,

$$S = \frac{g}{2}\int dx\ (\partial_{x^0} - i\partial_{x^1})\phi(x)(\partial_{x^0} + i\partial_{x^1})\phi(x) \tag{2.10.3}$$

where $x = (x^0, x^1)$ and $dx = dx^0 dx^1$. For making the above equation quadratic in fields we can use differentials and write the action,

$$S = \frac{1}{2}\int dxdy\ \phi(x)A(x,y)\phi(y) \tag{2.10.4}$$

where $A(x, y) = -g\delta(x - y)\partial_x^2$.

2.10.2 Correlation functions and OPEs

Since the action is now quadratic in fields we can use the theory of generating functional to quantise the free Bosonic fields. The generating functional is written as,

$$Z[J] = Z[0]e^{\frac{1}{2}\int dxdy\ J(x)K(x,y)J(y)} \equiv \int \mathcal{D}\phi\ e^{-S+\int dx J(x)\phi(x)} \tag{2.10.5}$$

where $K(x, y)$ is the Green's function for a differential operator and hence we get,

$$-g\partial_x^2 K(x, y) = \delta(x - y). \tag{2.10.6}$$

The solution for the above differential equation is given by,

$$K(x, y) = -\frac{1}{2\pi g}\log(\sqrt{(x - y)^2}). \tag{2.10.7}$$

The two point correlation functions can be derived using generating functional as follows:

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z[J]\Big|_{J=0} \tag{2.10.8}$$

If we use polar coordinates, the above equation generates Green's function.

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = K(r) \tag{2.10.9}$$

Therefore the two point correlation functions in polar coordinates is,

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = -\frac{1}{2\pi g}\log(r) = -\frac{1}{4\pi g}\log(r^2). \tag{2.10.10}$$

We can analytically continue the above generated correlation functions by complexifying Euclidean space. Define $z = x^0 + ix^1$ and $w = y^0 + iy^1$. The correlation functions, seperating out the holomorphic and anti-holomorphic parts become,

$$\langle\hat{\phi}(z)\hat{\phi}(w)\rangle = -\frac{1}{4\pi g}\log((\bar{z} - \bar{w})(z - w)) = -\frac{1}{4\pi g}(\log(z - w) + \log(\bar{z} - \bar{w})). \tag{2.10.11}$$

Comparing the above equation with equation (2.8.20), it is clear that the field for a mass-less free Boson does not behave as a primary field and hence we cannot predict whether the quantum field theory for massless free Boson follows conformal symmetry.

However if we take the derivative of the field $\phi(z)$, i.e. $\partial_z\phi(z)$ does behave like the primary fields. The two-point correlation then function becomes,

$$\langle\partial_z\hat{\phi}(z)\partial_w\hat{\phi}(w)\rangle = -\frac{1}{4\pi g}\frac{1}{(z - w)^2} \tag{2.10.12}$$

$$\langle\partial_{\bar{z}}\hat{\phi}(\bar{z})\partial_{\bar{w}}\hat{\phi}(\bar{w})\rangle = -\frac{1}{4\pi g}\frac{1}{(\bar{z} - \bar{w})^2} \tag{2.10.13}$$

OPE of $\partial_z \phi(z)$ with itself

The OPE for such a primary field with itself is then given by,

$$\partial_z \hat{\phi}(z) \partial_w \hat{\phi}(w) \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2)}. \tag{2.10.14}$$

The OPE is comparable to equation (2.8.20) and hence we conclude that the term $\partial_z \phi(z)$ is a primary field. However to make the CFT for such a field we need to find its OPE with stress-energy tensor.

OPE of $\partial_w \phi(w)$ and $\hat{T}(z)$

In complex coordinates the stress energy tensor for massless free boson can be derived using Noether's theorem for Klein-Gordon field. The stress energy tensor then becomes,

$$\hat{T}(z) = -2\pi g \boldsymbol{:} \partial_z \hat{\phi}(z) \partial_z \hat{\phi}(z) \boldsymbol{:}. \tag{2.10.15}$$

Note that the symbol $\boldsymbol{:}$ ensures the normal ordering of field operators. Putting the product of $\partial_w \phi(w)$ and $\hat{T}(z)$ inside an n-point correlation functions, computing its singular behaviour using contour integrals as $z \to w$ and dropping the regular functions for short distance approximation we can form OPE. Using the Wick's theorem we can write such an OPE as,

$$\hat{T}(z) \partial_w \hat{\phi}(w) \sim \frac{1}{(z-w)^2} \partial_w \hat{\phi}(w) + \frac{1}{z-w} \partial_w^2 \hat{\phi}(w). \tag{2.10.16}$$

Comparing the above equation with equation 9.11 with $h = 1$, we conclude that the primary field for the free Boson i.e. $\partial_z \phi(z)$ is a conformal field.

In this section we dealt with the conformal theory for a free Bosonic field. This is a fundamental example for a CFT. We encountered with quantum field which itself is not a primary field. We will use some aspects of the theory described in this section to develop the Lie algebra for a CFT.

2.11 The central charge and Virasoro Algebra

In this section we will briefly describe a very important aspect for a CFT known as central charge. Also we will use the OPEs formed in previous few sections to construct the Lie algebra of a CFT.

2.11.1 Descendant fields

In Section 2.10 we dealt with free Bosonic field which itself does not behave as a conformal field. There are many such examples. All fields don't follow the properties of a primary field. These fields don't obey the identity given by equation (2.9.9). Such fields often, are found to have higher than the double pole singularities. These are known as secondary fields.
A secondary field is any field that has higher than the double pole singularity in its operator product expansion with T and $\bar{T}$.
In general for any conformal field theory, the fields can be grouped into families $[\phi_n]$ each of which contains a single primary field ϕ_n and an infinite set of secondary fields (including its derivative), called its descendants. The set of all fields in a CFT $\{A_i\} = \Sigma_n \phi_n$ may be composed of either a finite or infinite number of conformal families.

2.11.2 Central charge

We compute the OPE of stress energy tensor for a free Bosonic field with itself. Using Wick's theorem and Taylor expansion we can write such an OPE as,

$$\hat{T}(z)\hat{T}(w) \sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w \hat{T}(w)}{z-w}. \tag{2.11.1}$$

The above OPE is exactly comparable to OPE of primary fields apart from the term quartic in denominator.
In general for any CFT the OPE of stress energy tensor with itself is given by,

$$\hat{T}(z)\hat{T}(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w \hat{T}(w)}{z-w}. \tag{2.11.2}$$

Here c is the termed as the central charge. It appears due to the central extension of Witt algebra. The central extension here refers to the fact that we allow $Conf(\mathbb{R}^{1,1})$ to act projectively rather than unitarily. Since two-point function is always positive central charge is always positive.
Note that for a mass-less free Boson $c = 1$.

2.11.3 Virasoro Algebra

We will now build the Lie Algebra of local conformal generators and find various Lie brackets. This algebra is termed as Virasoro algebra.
Making mode expansions of stress-energy tensor using Laurent series in an annular region in complex plane we get,

$$\hat{T}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n \tag{2.11.3}$$

$$\hat{\bar{T}}(\bar{z}) \equiv \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\hat{\bar{L}}_n. \tag{2.11.4}$$

Invert these equations by multiplying z^{n+1}, integrating around a circle, and applying the residue theorem

$$\hat{L}_n = \oint \frac{dz}{2\pi i}\ z^{n+1}\hat{T}(z) \tag{2.11.5}$$

$$\hat{\bar{L}}_n = \oint \frac{d\bar{z}}{2\pi i}\ \bar{z}^{n+1}\hat{\bar{T}}(\bar{z}). \tag{2.11.6}$$

Commutator brackets

Using the contour integrals described in 2.9 we can form the commutator bracket of two field operators $\hat{A}$ and $\hat{B}$. Instead of taking the difference of two contour integrals we will integrate a contour z around point w as shown in the Figure 3.5 . The commutator is thus given by,

$$[\hat{A}, \hat{B}] = \oint_0 dw \oint_W dz\ \hat{a}(z)\hat{b}(w) \tag{2.11.7}$$

where $\hat{A} = \oint dz\, \hat{a}(z)$ and $\hat{B} = \oint dz\, \hat{b}(z)$.

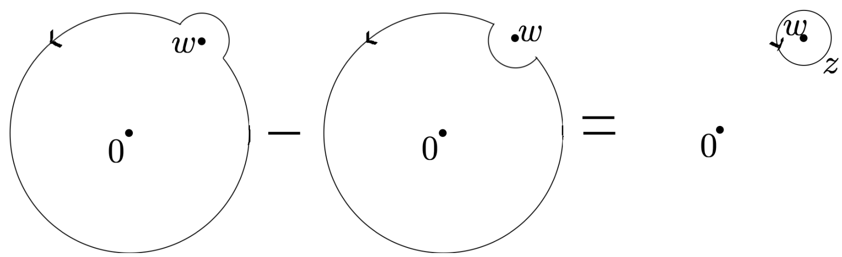

Figure 2.6: Contour integrals for solving Lie brackets

Using the mode expansion and OPEs we can further write,

$$[\hat{L}_n, \hat{L}_m] = \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{m+1} \oint_W dz\, z^{n+1} \left(\frac{c}{2} \frac{1}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w \hat{T}(w)}{z-w} \right) \tag{2.11.8}$$

Further using residue calculus we can solve the above equation and derive,

$$[\hat{L}_n, \hat{L}_m] = \frac{c}{12} n(n^2-1)\delta_{m+n,0} + (-m+n)\hat{L}_{m+n}. \tag{2.11.9}$$

Similarly we can derive for the anti-holomorphic part. The cross commutators are zero. The delta function term in the commutator relation is the phase factor of central extension of Lie albegra due to projective unitary representation. Therefore local conformal transformation forms a infinite dimensional Lie algebra which is termed as Virasoro algebra.

In this section we extended the OPEs constructed in Section 2.10 to define central charge. The central charge depends on the specific model which is studied using CFT. Further we used contour integrals to form the Virasoro algebra. Now that we have formed the Lie algebra of CFT we can form inner products and define the Hilbert space for a CFT.

2.12 Hilbert Space of CFT

A Hilbert space $\mathcal{H}$ can be defined as a vector space $\mathbb{H}$ containing inner products $\langle | \rangle$ and which is complete with respect to $\langle | \rangle$.
Using the generators of Virasoro algebra, defined in Section 2.11 we can construct the Hilbert space of any CFT. We will use the OPEs and correlation functions to form an inner product. Finally we will construct a subspace of Hilbert space rather than the whole vector space. In the end we look at the importance of central charge to construct a CFT.

2.12.1 Wightman Functions

In previous sections we defined the n-point correlation functions for a CFT using path integral formulation for primary fields. These correlation functions can be used to define the bra and ket vectors which can further be used to form inner products. Hence using correlation functions we can form Hilbert space of a CFT. We will use Wightman functions to build the Hilbert space. The Wightman functions are defined by,

$$w_n \equiv \langle 0| \hat{\phi}_1(x_1) \dots \hat{\phi}(x_n) |0\rangle . \tag{2.12.1}$$

We define ket vector by using fields operating on vaccum state, $|0\rangle$:

$$|x_1 \dots x_n\rangle \equiv \hat{\phi}(x_1) \dots \hat{\phi}(x_n) |0\rangle \tag{2.12.2}$$

Assuming that the field vectors are self-adjoint, the inner product will be defined by,

$$\langle x_1 \dots x_n | y_1 \dots y_m\rangle = \langle 0| \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \hat{\phi}(y_1) \dots \hat{\phi}(y_m) |0\rangle . \tag{2.12.3}$$

Here it is clear that Wightman functions contains smeared out versions of operators products. If we define f and g as smearing functions, then the inner products can be evaluated using,

$$\langle f_1 \dots f_n | g_1 \dots g_m\rangle = \int dx_1 \dots dx_n dy_1 \dots dy_m \bar{f}(x_1) \dots \bar{f}(x_n) g(y_1) \dots g(y_m) w(x_1, \dots, y_m) \tag{2.12.4}$$

2.12.2 States and Operators in CFT

Considering a radially quantised CFT we can define two states termed as "in" and "out" states with respect to the time, σ^0 flowing forwards or backwards. Let $\hat{A}(z,\bar{z})$ be a field associated to the "in" state. Such an "in" state is defined by,

$$|A_{in}\rangle \equiv \lim_{\sigma^0 \to -\infty} \hat{A}(\sigma^0, \sigma^1)\,|0\rangle \tag{2.12.5}$$

$$= \lim_{\sigma^0 \to -\infty} e^{\sigma^0 \hat{H}} \hat{A}(0, \sigma^1)\,|0\rangle \tag{2.12.6}$$

$$\tag{2.12.7}$$

Since we compactified space (to a Riemann sphere) all points at infinity correspond to the same point and we can create a state depending on field operator:

$$|A_{in}\rangle = \lim_{z,\bar{z} \to 0} \hat{A}(z,\bar{z})\,|0\rangle\,. \tag{2.12.8}$$

Similarly defining an "out" state dual to the "in" state we can write,

$$\langle A_{out}| \equiv \lim_{w,\bar{w} \to 0} \langle 0|\,\hat{\tilde{A}}(w,\bar{w}) \tag{2.12.9}$$

where $w = \frac{1}{z}$ and the operator $\hat{\tilde{A}}$ refers to a conformally transformed field operator.

Now for a primary field ϕ we know that under a conformal transformation:

$$f:\quad \hat{\phi}(w,\bar{w}) \to \hat{\phi}(f(w),\bar{f}(\bar{w}))(\partial_w f(w))^h(\partial_{\bar{w}}\bar{f}(\bar{w}))^{\bar{h}}. \tag{2.12.10}$$

Set $\hat{\tilde{A}} = \hat{\phi}$, the "out" state therefore becomes,

$$\langle \phi_{out}| = \lim_{z,\bar{z}\to 0} \langle 0|\,\hat{\phi}\left(\frac{1}{z},\frac{1}{\bar{z}}\right)\frac{1}{z^{2h}}\frac{1}{\bar{z}^{2\bar{h}}} \tag{2.12.11}$$

$$\equiv \lim_{z,\bar{z}\to 0} \langle 0|\,\hat{\phi}(\bar{z},z)^\dagger \tag{2.12.12}$$

$$= \left(\lim_{z,\bar{z}\to 0} \hat{\phi}(\bar{z},z)\,|0\rangle\right)^\dagger \tag{2.12.13}$$

$$\langle\phi|_{out} = (|\phi_{in}\rangle)^\dagger \tag{2.12.14}$$

where $|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0}\hat{\phi}(z,\bar{z})\,|0\rangle$

2.12.3 Finite Inner Products

Using the "out" and "in" states we can evaluate the inner products using Wightman functions:

$$\langle\phi_{out}|\phi_{in}\rangle = \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\,\hat{\phi}(z,\bar{z})^{\dagger}\hat{\phi}(w,\bar{w})\,|0\rangle \tag{2.12.15}$$

$$= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\,\hat{\phi}\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\hat{\phi}(w,\bar{w})\,|0\rangle \tag{2.12.16}$$

$$\tag{2.12.17}$$

Set $\xi = \frac{1}{z}$ and $\bar{\xi} = \frac{1}{\bar{z}}$. By compactifying space we can send $w, \bar{w} \to 0$. Hence the inner product becomes:

$$\langle\phi_{out}|\phi_{in}\rangle = \lim_{\xi,\bar{\xi}\to 0} \bar{\xi}^{2h}\xi^{2\bar{h}} \langle 0|\,\hat{\phi}(\bar{\xi},\xi)\hat{\phi}(0,0)\,|0\rangle \tag{2.12.18}$$

We are familiar with the expectation values since we derived such correlation functions in Section 2.5. Using Wightman functions and the correlation functions we can thus evaluate the inner product which would be a space-time independent quantity.

2.12.4 Building Hilbert Space for a CFT

We can form enough vectors to build $\mathcal{H}$ by acting, the operators used in Virasoro algebra, on vacuum state $|0\rangle$. The generators $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ close under a Lie algebra and thus belong to a global conformal transformation. If $\hat{T}(z)\,|0\rangle$ and $\hat{\bar{T}}(z)\,|0\rangle$ are regular as $z, \bar{z} \to 0$ we can show that the operators acting on vacuum state is invariant and hence,

$$\hat{L}_{-1}\,|0\rangle = \hat{L}_0\,|0\rangle = \hat{L}_1\,|0\rangle = 0. \tag{2.12.19}$$

Hence it is clear that $|0\rangle$ can be used to form the eigenstates of CFT.

Eigenstates of CFT

We will use OPE i.e. equation 9.11 to show that "in" states are eigenstates of Hamiltonian, $\hat{H} = \hat{L}_0 + \hat{\bar{L}}_0$. We know that the commutator relation is given by,

$$[\hat{L}_n, \hat{\phi}(w,\bar{w})] = \frac{1}{2\pi i}\oint_W dz\ z^{n+1}\hat{T}(z)\hat{\phi}(w,\bar{w}) \tag{2.12.20}$$

Using the OPE of $\hat{T}(z)\hat{\phi}(w,\bar{w})$ from equation 9.11 we can write,

$$[\hat{L}_n, \hat{\phi}(w,\bar{w})] = h(n+1)w^n\hat{\phi}(w,\bar{w}) + w^{n+1}\partial_w\hat{\phi}(w,\bar{w}). \tag{2.12.21}$$

By using the conformal weights h and $\bar{h}$ we can define asymptotic "in" state as:

$$|h,\bar{h}\rangle \equiv \hat{\phi}(0,\bar{0})\,|0\rangle\,. \tag{2.12.22}$$

Using equation 2.12.22 we can prove the following identities,

$$\hat{L}_0\,|h,\bar{h}\rangle = h\,|h,\bar{h}\rangle\,. \tag{2.12.23}$$

$$\hat{\bar{L}}_0\,|h,\bar{h}\rangle = \bar{h}\,|h,\bar{h}\rangle\,. \tag{2.12.24}$$

$$\bar{L}_n\,|h,\bar{h}\rangle = \hat{\bar{L}}_n\,|h,\bar{h}\rangle = 0, \forall\ n > 0. \tag{2.12.25}$$

Hence the asymptotic "in" state seems to be an eigenstate of the Hamiltonian. Excited states can be created by using mode expansion of primary field as a Laurent series given by,

$$\hat{\phi}(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m+h}\bar{z}^{-n+\bar{h}}\hat{\phi}_{m,n}. \tag{2.12.26}$$

The modes of primary fields are given by inverting the above equation:

$$\hat{\phi}_{m,n} = \left(\frac{1}{2\pi i}\oint dz\, z^{m+h+1}\right)\left(\frac{1}{2\pi i}\oint d\bar{z}\,\bar{z}^{n+\bar{h}+1}\right)\hat{\phi}(z,\bar{z}). \tag{2.12.27}$$

Note that for the "in" and "out" states to be well defined, it is required that the modes annihilate the vacuum state:

$$\hat{\phi}_{m,n}\,|0\rangle = 0, \forall\ m > -h \text{ and } n > -\bar{h}. \tag{2.12.28}$$

We can drop the dependence on $\bar{z}$ and define the holomorphic mode as,

$$\hat{\phi}_n \equiv \frac{1}{2\pi i}\oint dz\, z^{n+h-1}\hat{\phi}(z). \tag{2.12.29}$$

Now using equation (2.12.22) we can derive:

$$[\hat{L}_n, \hat{\phi}_m] = (n(h-1)-m)\hat{\phi}_{m+n}. \tag{2.12.30}$$

These operators therefore act like ladder (raising or lower) operators.
Note that $\bar{L}_{-m}$ raises the conformal dimension:

$$[\hat{L}_0, \hat{L}_{-m}] = m\hat{L}_{-m}. \tag{2.12.31}$$

2.12.5 Verma modules

Instead of building the whole Hilbert space we focus on the subspace of $\mathcal{H}$ constructed by applying $\hat{L}_n$ on the primary fields. Define an "in" state by $|h, \bar{h}\rangle = \hat{\phi}(0, \bar{0})\,|0\rangle$
For negative indices define descendant states as:

$$\hat{L}_{-k_1} \dots \hat{L}_{-k_n}\,|h\rangle\,, \text{where } 1 \le k_1 \le \dots \le k_n. \tag{2.12.32}$$

The descendant state is an eigenvector of $\bar{L}_0$ and the eigenvalue is given by, $h' = h + k_1 + \dots + k_n \equiv h + N$, where we call $N = \sum_{j=1}^{n} k_j$ the Level.
The subspace of full Hilbert space is generated by descendant states and is known as module. Under the operation of Virasoro generator to these descendant states the module serves as a representation of Virasoro algebra. The module formed by asymptotic "in" states is called Verma module.

2.12.6 Central charge and Hilbert space

Using Virasoro algebra we can derive the following relations:

$$\hat{L}_n^\dagger = \hat{L}_{-n}, \tag{2.12.33}$$

$$\hat{L}_0\,|h\rangle = h\,|h\rangle\,, \tag{2.12.34}$$

$$\hat{L}_n\,|h\rangle = 0, \forall\ n > 0, \tag{2.12.35}$$

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}, \tag{2.12.36}$$

From these relations we can derive:

$$\langle h|\,[\hat{L}_{-n}^\dagger, \hat{L}_{-n}]\,|h\rangle = \Big(2nh + \frac{c}{12}(n^3 - n)\Big)\,\langle h|h\rangle\,. \tag{2.12.37}$$

- If $n = 1$,the equation would generate positive values iff $h \ge 0$.
  - If $h = 0$ for arbitary c we get vaccum state expectation value.
  - If $h = 0$ and $c = 0$ we get a trivial representation.
- If $c = 0$ and arbitrary h we consider the Gram matrix determinant given by:

  $$\det(\text{Gram}) = 4n^3 h^2 (4h - 5n). \tag{2.12.38}$$

  If $n \to \infty$ for $c = 0$ we get negative determinant.

- For $n \to \infty$ and $c > 0$ from the above equation we get non-trivial representations of Virasoro algebra.

Hence for any case the expectation value is valid only if $c > 0$. Therefore we can form a Hilbert space and hence a CFT only if central charge exists and has a positive value.

In this section we focused on steps to build a Hilbert space for a CFT. Instead of constructing the whole Hilbert space itself we attempted to construct a subspace known as Verma module using Virasoro algebra. Also we saw the paramount importance of the central charge to construct a Hilbert space.

2.13 Concluding Remarks

Before switching over to the next model in this thesis, let us take note of some important features of the analysis of conformal field theory.

- It is quite clear that $d = 2$ is very important for studying CFT. The main reason, being the existence of infinite local symmetries and hence much more constraints in the case of a CFT in $d = 2$.
- Correlation functions serve as the constraints on a CFT giving us a criterion to identify a conformally invariant QFT.
- Ward identities help us to realise the generators of conformal symmetry.
- Radial Quantisation lets us define a CFT on a complex plane.
- Virasoro algebra corresponds to the Lie algebra of a conformally invariant QFT.

Chapter 3

Hawking Information Paradox

3.1 The "Quantum" Black Holes

One can infer from Einstein's general theory of relativity, that if matter is sufficiently compressed, the gravity can become extremely strong. These regions of spacetime do not let any object to escape out, not even light. The region thus formed is known as black hole. The boundary of a black hole is termed as an event horizon; objects can fall in from the horizon but can never escape. The present state of universe grants us observations of such regions when stellar mass collapses. The black hole thus formed is known as stellar black hole. Such black holes can be studied using classical theory of general relativity.

However the early universe is said to be composed of smaller black holes, where one cannot neglect the quantum effects of black holes. Stephen Hawking demonstrated some astonishing works by providing us with a profound theory for studying these black holes [9]. The theory unifies three very different branches of physics namely; general relativity, quantum theory and thermodynamics. Since quantum mechanics plays a very crucial role in studying the physics of these black holes, we can term them as "Quantum" black holes. Unlike the classical theory of black holes, quantum theory affirms the fact that black holes do radiate once they start shrinking. The evaporated radiation is termed as Hawking radiation. The black holes can therefore be believed as thermally radiating objects. One can then study black hole thermodynamics.

AdS/CFT correspondence principle plays an important role while studying

quantum aspects of black holes. The correspondence principle relates two theories viz., a thoery of quantum gravity in Anti-de Sitter(AdS) spacetime and a conformal field theory(CFT) being dual to each other [13]. The CFT can be imagined to be living on the boundary of AdS. This duality serves as a vital step when we want to study a black hole system in an AdS, where we can place the black hole in an AdS spacetime and study the properties of system by studying dynamics of boundary CFTs. We employ this technique to study the case of eternal black hole in AdS. [14]
A detailed derivation of Hawking radiation is reviewed by considering the cases of Rindler spacetime and Unruh effect. Further we review the qualitative analysis of information paradox and the attempts to resolve it, employing the use of quantum extremal surfaces.[2]

3.2 AdS Spacetime

Any relativistic system must be analysed by embedding it into some spacetime structure. The study of Black Hole thermodynamics becomes easier as well as neat if we embed the system into an Anti-de-Sitter(AdS) spacetime. In this section we describe important features of such AdS spacetimes.
Any spacetime structure with constant positive curvature is defined to be a de-Sitter(dS) spacetime. Hence we define AdS spacetime as a structure having constant negative curvature.

3.2.1 Spaces with constant curvature

Before discussing spacetimes with constant negative curvatures, it is worthwhile to briefly study spaces with constant curvature. We will consider simple examples of embedding systems in such spaces.

As a first example consider Euclidean space, the metric for such a space can be written as,

$$ds^2 = dX^2 + dY^2 + dZ^2 \tag{3.2.1}$$

Now consider a sphere S^2, such a surface is defined by,

$$X^2 + Y^2 + Z^2 = L^2 \tag{3.2.2}$$

The spherical polar coordinates are given by,

$$X = Lsin\theta cos\phi \tag{3.2.3}$$

$$Y = Lsin\theta sin\phi \tag{3.2.4}$$

$$Z = Lcos\theta \tag{3.2.5}$$

Using these coordinates the metric equation becomes,

$$ds^2 = L^2(d\theta^2 + sin^2 d\phi^2) \tag{3.2.6}$$

SO(3) transformation maps any point on S^2 to other points on S^2. Hence S^2 is homogeneous and respects SO(3) invariance of ambient-Euclidean space. Therefore S^2 is embedded in Euclidean space.
Next consider hyperbolic space H^2. H^2 is described by,

$$-Z^2 + X^2 + Y^2 = -L^2 \tag{3.2.7}$$

It is clear that H^2 cannot be embedded into Euclidean space. However a Minkowski space is given by,

$$ds^2 = -dX^2 + dY^2 + dZ^2 \tag{3.2.8}$$

H^2 respects SO(1,2) invariance of ambient-Minkowski space. H^2 can hence be embedded into 3-D Minkowski space. Note that H^2 is not a spacetime but a space embedded into Minkowski spacetime.

3.2.2 Spacetimes with constant curvature

Let us now analyse embedding spacetime structures rather than just spaces. A de-Sitter spacetime is described by,

$$-Z^2 + X^2 + Y^2 = L^2 \tag{3.2.9}$$

A dS_2 spacetime can be embedded into Minkowski spacetime having one timelike direction,

$$ds^2 = -dX^2 + dY^2 + dZ^2 \tag{3.2.10}$$

dS_2 has therefore SO(1,2) invariance.
Next consider AdS_2 spacetime given by,

$$-Z^2 - X^2 + Y^2 = -L^2 \tag{3.2.11}$$

AdS_2 can be embedded into Minkowski spacetime with two timelike directions,

$$ds^2 = -dZ^2 - dX^2 + dY^2 \tag{3.2.12}$$

For embedding AdS_2 we use coordinate transformation,

$$Z = L\cosh\rho\cos\widetilde{t} \tag{3.2.13}$$

$$X = L\cosh\rho\sin\widetilde{t} \tag{3.2.14}$$

$$Y = L\sinh\rho \tag{3.2.15}$$

Using these coordinates the metric becomes,

$$ds^2 = L^2(-\cosh^2\rho d\tilde{t} + d\rho^2) \tag{3.2.16}$$

Here ($\tilde{t}$,ρ) are referred to as global coordinates of AdS spacetime. AdS_2 spacetime therefore has SO(2,1) invariance. Note that although we embed AdS_2 spacetime in two timelike coordinates, AdS_2 spacetime itself has only one timelike coordinate. Figure 3.1 shows embedding of AdS_2 in Minkowski spacetime.

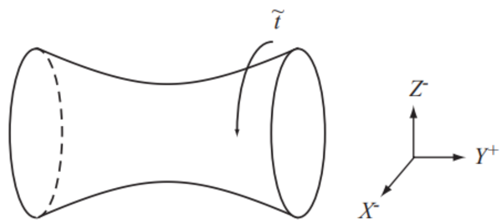

Figure 3.1: The embedding of AdS_2

3.2.3 Coordinate systems of AdS spacetime

We now mention a few coordinate systems used to analyse AdS spacetime.

- Global coordinates: $(\tilde{t},\rho)$

$$\frac{ds^2}{L^2} = (-\cosh^2\rho d\tilde{t}^2 + d\rho^2) \tag{3.2.17}$$

- Static coordinates: $(\tilde{t},\tilde{r})$ where $\tilde{r} := \sinh\rho$

$$\frac{ds^2}{L^2} = -(\tilde{r}^2+1)d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2+1} \tag{3.2.18}$$

- Conformal coordinates: $(\tilde{t},\theta)$ where $\tan\theta := \sinh\rho$

$$\frac{ds^2}{L^2} = \frac{1}{\cos^2\theta}(-d\tilde{t}^2 + d\theta^2) \tag{3.2.19}$$

  Note that there exists a boundary ($\theta = \pm\pi/2$). This boundary can be used to define guage theories in AdS/CFT duality.

- Poincare coordinates: (t,r) The coordinate transformation becomes,

$$Z = \frac{Lr}{2}(-t^2 + \frac{1}{r^2} + 1) \tag{3.2.20}$$

$$X = Lrt \tag{3.2.21}$$

$$Y = \frac{Lr}{2}(-t^2 + \frac{1}{r^2} - 1) \tag{3.2.22}$$

The metric in Poincare coordinates is given by,

$$\frac{ds^2}{L^2} = -r^2 dt^2 + \frac{dr^2}{r^2} \tag{3.2.23}$$

Again there exists a boundary at $r \to \infty$.

It is clear that conformal coordinates and Poincare coordinates become important while studying black hole systems in AdS. Figure 3.2 shows an AdS_2 spacetime in conformal coordinates. If we use Poincare coordinates it is clear that, Poincare coordinates cover only a part of the full AdS spacetime as shown by dark shaded region. Such a region is known as Poincare patch. The remnants of this section will come in handy while describing evaporation

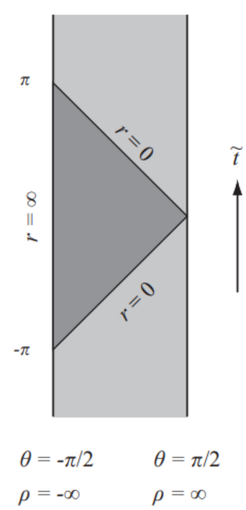

Figure 3.2: AdS_2 using conformal coordinates. The dark region corresponds to Poincare patch.

of Black Holes. This will in turn help us to study Hawking process and hence to analyse the information paradox.

3.3 Black Hole thermodynamics

In this section we aim to briefly study thermodynamic properties of black holes. Also we will roughly discuss the mathematical analysis of Hawking Radiation.

3.3.1 Schwarzschild Black Hole

In general relativity, Einstein field equation is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{3.3.1}$$

where G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor of matter fields. Here $R_{\mu\nu}$ is the Ricci curvature tensor, while R is curvature scalar. $g_{\mu\nu}$ is the metric of spacetime. Einstein field equation claims that energy momentum tensor of matter fields determines the spacetime curvature. Solving this equation can give us solutions for black hole systems. We now look at the simplest solution of this equation by setting,

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$$

where Λ is called the cosmological constant. Now the field equation becomes,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{3.3.2}$$

If we set $\Lambda = 0$ we get,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{3.3.3}$$

The solution of this equation gives the simplest solution of black hole, Schwarzschild solution. The metric of Schwarzchild black hole is given by,

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2 \tag{3.3.4}$$

Using the above metric we can derive the Schwarzschild radius of black hole horizon, given by,

$$r_0 = 2GM/c^2$$

3.3.2 Black Hole Thermodynamics: An Analogy

Zeroth Law

Any thermodynamic system eventually reaches a thermal equilibrium and the temperature reaches a constant value everywhere. This is the zeroth law of thermodynamics.
It can be shown that any asymmetric black hole eventually evolves into a spherically symmetric one. We can therefore draw an analogy between thermal equilibrium and spherical symmetry. It can also be shown that the surface gravity of a spherically symmetric black hole is constant. This is analogous to zeroth law and surface gravity plays the role of temperature.

Second Law

The black hole being spherically symmetric, we can write the area of black hole horizon as, $A = 4\pi r_0^2$. Hence

$$A = \frac{16\pi G^2 M^2}{c^4} \tag{3.3.5}$$

The horizon being proportional to mass of the black hole implies that matter falling into black hole tends to increase the area of the horizon. On the other hand classical relativistic results suggest that nothing can escape from the black hole. Hence area seems to be a never decreasing quantity.
Thermodynamical entropy is a never decreasing quantity. As a first intuition we can therefore associate the area of the horizon to entropy of black hole. Therefore an ever increasing area of black hole horizon can be stated as second law.

First Law

If we differentiate equation (3.3.5) we obtain,

$$dA = \frac{16\pi G^2 2M}{c^4} dM$$

Rearranging the above equation we obtain,

$$GdM = \frac{\kappa}{8\pi} dA \tag{3.3.6}$$

where $\kappa = c^4/4GM$ is the surface gravity.

The first law of thermodynamics is $dE = TdS$. Since area of horizon is a measure of entropy, we can call equation (3.3.6) as the first law.

3.3.3 Enter AdS/CFT

So far we have crudely analysed the black hole analogous to the laws of thermodynamics. However there are a few problems with our analysis, the important ones being:

- The analogy suggests that black holes behave similar to a thermal body. Hence black holes must radiate! However classical study of black holes suggests that, black holes are bodies which never let anything to escape.
- Black holes are made from ordinary matter, hence they must obey rules of quantum mechanics. Also, infalling matter must be following rules of quantum mechanics.

The above two statements precisely demand that we must treat the theory of black holes using quantum mechanics. Once we do this we will be familiar with Hawking radiation. The detailed quantum mechanical study of Hawking radiation is given in the forthcoming sections. In this section we crudely analyse the relation between Hawking radiation and laws of thermodynamics for black hole.
Considering quantum effects of matter, we can show that the black hole indeed emits the black body radiation known as the Hawking radiation and its temperature is given by,

$$k_b T = \frac{\hbar c^3}{8\pi G M} \tag{3.3.7}$$

Using the above equation and equation (3.3.5) one can show that,

$$d(Mc^2) = \frac{T k_b c^3}{4G\hbar} dA \tag{3.3.8}$$

Comparing with $dE = TdS$ one obtains;

$$S = \frac{1}{4}\frac{A}{\ell_{pl}^2} k_b \tag{3.3.9}$$

where $\ell_{pl} = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length.

Note that the above expression suggests that black hole entropy is proportional to the area while we know that the statistical entropy of any system is proportional to the volume. This suggests that a black hole can be described by the usual statistical system whose spatial dimension is one dimension lower than the gravitational theory. This serves as the first clue of employing AdS/CFT to study black hole thermodynamics. It will become more clear when we study Eternal Black holes in upcoming sections.

3.4 Eternal Black Holes in AdS

In 3.3.1 we derived the Schwarzschild solution for Einstein field equation. It is clear that such a solution is a steady state solution of the field equation. Hence Schwarzschild black hole appears to one as being in the same state and does not evolve. It seems to have been existing since forever and will exist till eternity. Such a maximally extended Schwarzschild solution of field equation is termed as Eternal Black Hole.
Study of Eternal black hole in Anti-de-Sitter spacetime using AdS/CFT will help us to construct a thorough formulation of Hawking radiation. Once we have a better understanding of Hawking radiation we can use the ideas to study black hole information paradox in upcoming sections.

3.4.1 Penrose diagram of Schwarzschild solution

The Schwarzschild black hole is given by metric,

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2 \tag{3.4.1}$$

It is clear that the metric has a curvature singularity at origin, $r = 0$. Also the metric has a singularity at $r = 2M$ which represents a breakdown of the coordinate system.
To study the behaviour of the metric near $r = 2M$, we introduce tortoise coordinate system,

$$r^* = r + 2Mln(\frac{r}{2M} - 1) \tag{3.4.2}$$

Hence we obtain $dr = (1 - 2M/r)dr^*$. Using these coordinate changes, the metric becomes;

$$ds^2 = -(1 - \frac{2GM}{r})(-dt^2 + dr^{*2}) + r^2(r^*)d\Omega_2^2 \tag{3.4.3}$$

Hence the null geodesics correspond to $t = \pm r^*$. We can form null coordinates;

$$u = t - r^* \tag{3.4.4}$$
$$v = t + r^*$$

Further we introduce the null Kruskal coordinates;

$$\bar{u} = -4Me^{-u/4M} \tag{3.4.5}$$
$$\bar{v} = 4Me^{v/4M}$$

The region $-\infty < r^* < \infty$ translates in Kruskal coordinates to $-\infty < \bar{u} < 0$, $0 < \bar{v} < \infty$. Also the metric transforms to;

$$ds^2 = -\frac{2M}{r}e^{-r/2M}d\bar{u}d\bar{v} + r^2 d\Omega_2^2 \tag{3.4.6}$$

Note that the metric is now non-singular at $r = 2M$. Therefore we can cover the entire region using Kruskal coordinates by analytically continuing to $-\infty < \bar{u}$ and $\bar{v} < \infty$.

Using Kruskal coordinates we can form a non-physical metric which might be conformal and introduce a further coordinate change to compactify the coordinates. The infinities will now correspond to some finite points in new coordinate system. The compactified diagram can then be referred to as Penrose diagram. Without dealing with the mathematical intricacies of the mentioned procedure, we directly present the Penrose diagram of Schwarzschild black hole given by Figure 3.3.

Note that in a Penrose diagram the null geodesics are always represented by 45° lines. It helps us in determining whether two points are in causal contact.

The construction of Penrose diagram plays an important role in studying Eternal black holes.

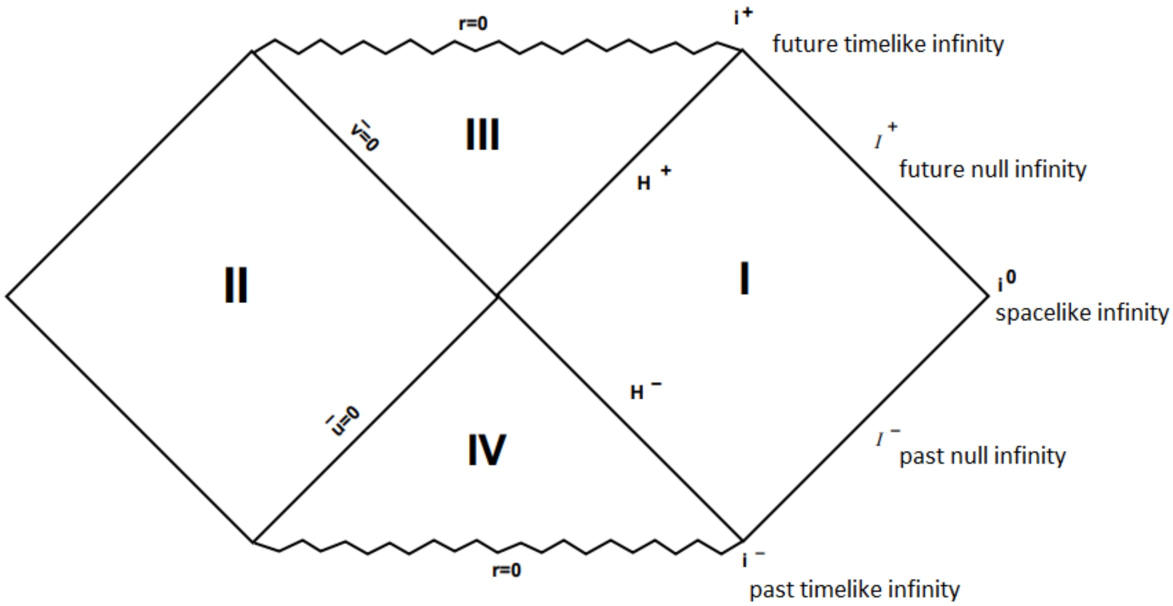

Figure 3.3: Penrose diagram of maximally extended Schwarzschild black hole

3.4.2 The Proposal

The Penrose diagram of maximally extended Schwarzschild black hole in AdS^{d+1} is given by Figure 3.4. We have four regions. Region I covers the region outside the horizon from the point of view of an observer on the right boundary. Region II is a copy of the region I and includes another boundary. Regions III and IV contain spacelike singularities. In the diagram over each point there is a sphere S^{d-1}.
Since region I and II are alike and contain two boundaries, we can use AdS/CFT correspondence and attach a CFT to both the boundaries. We will propose that the spacetime can be holographically described by considering two identical and non-interacting copies of the conformal field theory[3]. However, such a proposal can be made only if we consider a particular entangled state of the two CFTs. According to the proposal it is clear that we have two models to study now, one being the eternal black hole itself placed in AdS^{d+1} and the other being the CFTs placed at the boundary.

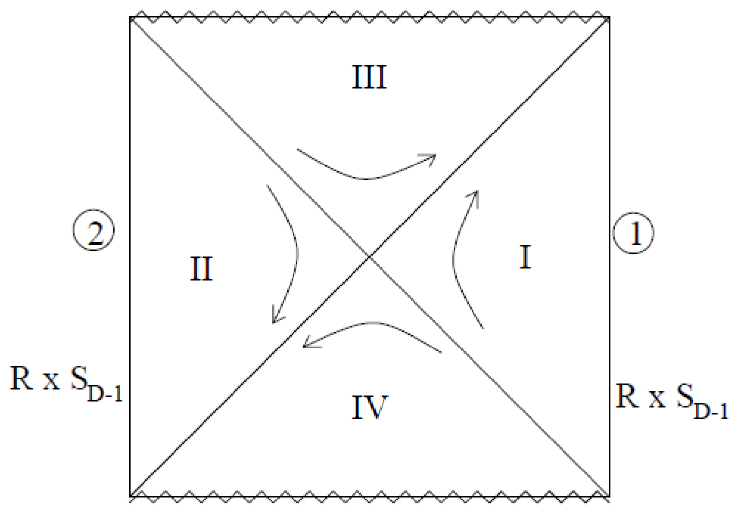

Figure 3.4: Penrose diagram for an Eternal Black Hole in $AdS_{d+1} spacetime$.

3.4.3 Description of the Black Hole system

We now give a detailed construction of Eternal Black holes in AdS^{d+1}. We consider black holes with positive specific heat so that the system gives a dominant contribution to the canonical thermodynamic ensemble. Let us now focus on AdS^3, we can later generalize the same for AdS^{d+1}. The Euclidean metric for three dimensional black hole can be written as;

$$ds^2 = (r^2 - 1)d\tilde{\tau}^2 + \frac{dr^2}{r^2 - 1} + r^2 d\tilde{\phi}^2 \qquad (3.4.7)$$

We retain the periodicity in space-time coordinates as : $\phi = \phi + 2\phi$; $\tau = \tau + \beta$. Here β can be identified as the inverse temperature of the CFT. We can further complexify the metric by transforming coordinates as:

$\tilde{\phi} = \frac{2\pi\phi}{\beta}$; $\tilde{\tau} = \frac{2\pi\tau}{\beta}$ and $z = |z|e^{i\tilde{\tau}}$.

The metric then changes to:

$$ds^2 = 4\frac{dzd\bar{z}}{(1-|z|^2)^2} + \frac{(1+|z|^2)^2}{(1-|z|^2)^2}d\tilde{\phi}^2. \tag{3.4.8}$$

By analytically continuing the metric in imaginary part of z we can set $z = -v$ and $\bar{z} = u$. This will in turn give us the metric in Kruskal coordinates:

$$ds^2 = -4\frac{dudv}{(1+uv)^2} + \frac{(1-uv)^2}{(1+uv)^2}d\tilde{\phi}^2. \tag{3.4.9}$$

Here $u = t + x$ and $v = t - x$. The eternal black hole in the Kruskal null coordinates is represented by Figure 3.5. The event horizons are at

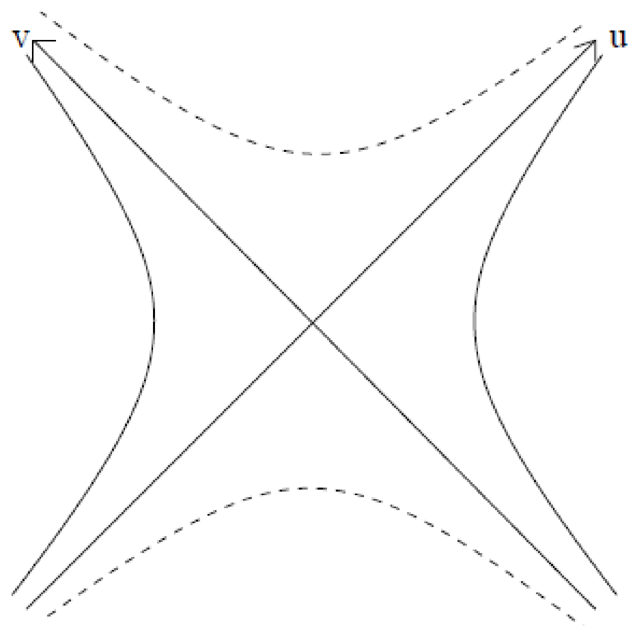

Figure 3.5: Lorrentzian black hole in Kruskal coordinates.

$u = 0$ and $v = 0$. The past and future singularities lie at $uv = 1$; while the boundary of AdS lies at $uv = -1$. In such a coordinate system we can evolve the wavefunction initially in Euclidean signature while later in Lorrentzian signature. This prescription of evaluating the wavefunction is known as Hartle-Hawking construction of the wavefunction.

3.5 An attempt to prepare Hartle-Hawking state

We now try to construct the Hartle-Hawking state using techniques of QFT. We will derive some important aspects of Hawking radiation manifested by properties of spacetime structure.

3.5.1 Rindler Space

Two dimensional Rindler space is a patch of Minkowski space. In 2-D the Rindler metric is given by:

$$ds^2 = dR^2 - R^2 d\eta^2 \tag{3.5.1}$$

There is a horizon at $R = 0$ and hence the metric is well behaved for $R > 0$. Now consider metric for $R^{1,1}$:

$$ds^2 = -dt^2 + dx^2 \tag{3.5.2}$$

If we perform a coordinate change such that: $x = R\cosh\eta$; $y = R\sinh\eta$ it is clear that the above metric changes to the Rindler metric. The coordinate change allows us to write: $x^2 - t^2 = R^2 > 0$. This implies that Rindler coordinates cover a patch of Minkowski space $R^{1,1}$ such that: $x > 0$ and $|t| < x$.

We define a Rindler observer as one sitting at fixed R. These observers are effectively confined to a piece of Minkowski space and can see a horizon at $R = 0$, similar to that of a black hole.

3.5.2 Near the black hole horizon

The region very near to the black hole horizon can be approximated to be a Rindler space. Consider the metric of Schwarzschild black hole:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{3.5.3}$$

where $f(r) = 1 - 2M/r$. Now we make the coordinate change:

$$r = 2M(1 + \epsilon^2) \tag{3.5.4}$$

Hence $f(r) \approx \epsilon^2$. If we consider ϵ to be small, the metric can be expanded as:

$$ds^2 = -\epsilon^2 dt^2 + 16M^2 d\epsilon^2 + 4M^2 d\Omega_2^2 + ... \tag{3.5.5}$$

The (t, ϵ) part of the above metric corresponds to Rindler space. Hence we can use Rindler metric for our construction of Hartle-Hawking state.

3.5.3 Unruh radiation

Consider Minkowski spacetime in vaccum state. An observer on worldline x will not observe any excitations and hence will record a zero temperature for such a vaccum state. Now consider a Rindler observer accelerating with acceleration a. This corresponds to a Rindler observer at $R = 1/a$.

No unique vaccum state

We now check whether the Rindler vacuum is same as the Minkowski vacuum. In QFT, there is no such unique state as a vacuum state, rather the so-called vacuum state is observer dependent. In QFT, any state is expanded as quantum fields in some energy mode. Such mode expansion involves two operators, a_ω and $a_\omega^\dagger$, one being the annihilation operator and other being the creation operator respectively. Now vacuum state is defined as $|0\rangle$ given by $a_\omega |0\rangle = 0$. Excitations are further created by positive energy modes using $a_\omega^\dagger$. The energy however in quantum mechanics is given by expectation value of Hamiltonian. But the Hamiltonian manifests itself as a time-evolution operator:

$$\frac{i}{\hbar}[H, O] = \partial_t O \tag{3.5.6}$$

Hence Hamiltonian depends on time coordinate t. Therefore different choices of time coordinate will correspond to different choices of Hamiltonian and hence different notions of vaccum state. Thus, one cannot agree over any unique vacuum state in QFT.

Unruh temperature

The Rindler coordinate η is periodic under $\eta \sim \eta + 2\pi i$. If inverse temperature is $\beta = 2\pi$ then the Rindler coordinate looks similar to a temperature $T = 2\pi$. Consider it as the temperature associated with time translation ∂_η. The proper time of a Rindler observer at $R = R_0$ is $d\tau = R_0 d\eta$. Hence the temperature actually observed would be:

$$T_{unruh} = \frac{1}{\sqrt{g_{\eta\eta}}}\frac{1}{2\pi} = \frac{1}{2\pi R_0} \tag{3.5.7}$$

In our case we had an accelerating observer $R_0 = 1/a$. Hence the Unruh temperature becomes:

$$T_{unruh} = \frac{a}{2\pi} \tag{3.5.8}$$

Hence it is clear that vacuum state in Minowski space corresponds to a radiating state in Rindler space.

Thus, it is clear that Unruh observers see a heat bath due to periodicity in imaginary time. Also near the black hole horizon, we can consider Rindler space. Hence we can confer the fact that black holes radiate like a blackbody at temperature T. This radiation is known as Hawking radiation.

3.5.4 Hartle-Hawking state using Rindler space

Now that we are familiar with the notion of Hawking radiation and Rindler coordinates we make an attempt to construct Hartle-Hawking state for Eternal black hole.

Consider a state analogous to vacuum state over a black hole background given by,

$$ds^2 = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{3.5.9}$$

with the imaginary time identification: $\tau \sim \tau + \beta$. The spacetime has no interior, $r > 0$ always. The spacetime is as shown in Figure 3.6.

The $t = 0$ slice in Lorrentzian spacetime corresponds to $\tau = 0$ slice in Euclidean spacetime. The Lorrentzian path integral will be calculated over a part of spacetime and hence Rindler coordinates can be used. The Euclidean path integral will produce a pure highly entangled state over two sided Lorrentzian spacetime. This highly entangled state describes eternal black hole in equlibrium with thermal bath of radiation outside the black hole. This state is known as Hartle-Hawking(HH) state.

The construction of H-H state in the manner prescribed as of yet i.e. by

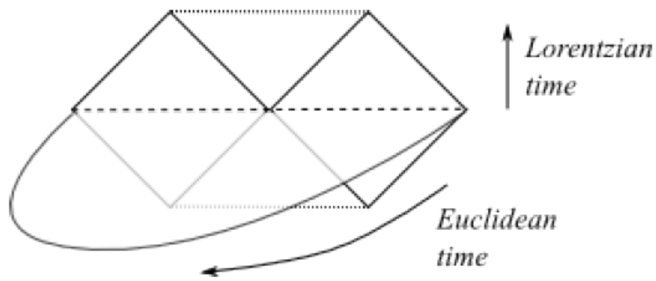

Figure 3.6: Spacetime for evolution of HH state

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using properties of spacetime seems to be difficult. However in the next section we will explore the construction of H-H state using the coupled CFTs to the Eternal black hole.

3.6 Hartle-Hawking state: Using CFT

We will now try to understand the wavefunction of Hartle-hawking state from the point of view of boundary CFT. The path integral will give us a wavefunction on the product of two copies of the CFT. Let $\mathcal{H} = \mathcal{H}_1 X \mathcal{H}_2$ denote the full Hilbert space consisting of two copies of the Hilbert space of the CFT. The wavefunction $|\psi\rangle \in \mathcal{H}$, representing the H-H state is:

$$|\psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} \left|E_n\right\rangle_1 X \left|E_n\right\rangle_2 \tag{3.6.1}$$

Here $Z(\beta)$ is the partition function of one copy of the CFT at inverse temperature β. The sum runs over all the energy eigenstates of the system, the index $1, 2$ indicate the Hilbert space of the corresponding state.

3.6.1 TFD

The construction of the above mentioned wavefunction can be studied using real time thermal field theories and is known as ThermoField Dynamics

(TFD). The state consisting of two field theories is known as thermofield double state. The analysis presents us with a nice trick of choosing one QFT and copying it and hence consider a new QFT which is actually a tensorial product over two original QFTs. In our case the QFT corresponds to the boundary CFTs, the states are thus the eigenstates of eacy copy and the wavefunction corresponds to the same as in equation (3.6.1).
Let us now define an operator $\mathcal{O}_1$ on the first copy of the CFT i.e. on $|E_n\rangle_1$. We can show that,

$$\langle\psi|\,\mathcal{O}_1\,|\psi\rangle = \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n^1}\,\langle E_n^1|\,\mathcal{O}_1\,|E_n^1\rangle \tag{3.6.2}$$

On the other hand it can be shown that:

$$Tr[\mathcal{O}_1\rho_\beta] = \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n^1}\,\langle E_n^1|\,\mathcal{O}_1\,|E_n^1\rangle \tag{3.6.3}$$

where Tr calculates the trace and we denote the density matrix ρ_β as:

$$\rho_\beta = \frac{e^{-\beta H}}{Tr[e^{-\beta H}]}$$

Hence it is clear that:

$$\langle\psi|\,\mathcal{O}_1\,|\psi\rangle = Tr[\mathcal{O}_1\rho_\beta] \tag{3.6.4}$$

The left hand side does not contain any operators acting on the second copy of CFT and hence we can trace out the sum over all states of the second copy and doing so will lend us the result on right hand side. Hence if we restrict ourselves only to the first state as a pure state of the system, it behaves like a thermal state. The procedure of tracing out the states is known as purifying the thermal state. Any mixed state can be purified by adding enough auxillary states(in our case copy of CFT) and tracing them out. Hence we can conclude that the wavefunction describing H-H state is a state of copies of two pure states.
Further the two pure states or copies being decoupled are correlated. If we define $Tr[\mathcal{O}_2]$ as acting on the second copy then it can be shown that $\langle\psi|\,\mathcal{O}_1\mathcal{O}_2\,|\psi\rangle$ is non zero. This can let us conclude that the two copies exist in an entangled state. H-H state is therefore, a maximally entangled state. Hence we can guess as a first intuition that Hawking radiation might emit pairs of entangled particles.

3.7 More on Black Hole Thermodynamics

We now move forward to postulate the information paradox and explore the possible resolution of the paradox. Section 3.3 dealt with the analogy of thermodynamical laws and the properties of a radiating black hole. We now formulate those laws in a more technical way.
When an object is dropped into a black hole, the event horizon ripples for a brief amount of time and then quickly attains new equilibrium at a larger radius. The same statement can be formulated for a rotating black hole as:

$$\frac{\kappa}{8\pi G_N} d(Area) = dM - \Omega dJ \tag{3.7.1}$$

Here κ:=Surface gravity of black hole
G_N:=Newton's gravitational constant
M:=Mass of black hole
Ω:=Rotational velocity of black hole horizon
J:=Angular momentum for the rotating black hole.
We postulate that black hole has temperature T proportional to the surface gravity κ of the black hole. Also we know that the black hole has a entropy S_{BH} proportional to the area of the black hole. Hence equation (3.7.1) can be written as:

$$TdS_{BH} = dM - \Omega dJ \tag{3.7.2}$$

This is the first law of thermodynamics for black hole.

3.7.1 Black hole entropy

We know from section 3.3 that the area of black hole horizon always increases, analogous to the second law of thermodynamics. In classical general relativity black holes can neither have a temperature nor any entropy. However from the analysis of eternal black holes, we showed by coupling general relativity to a QFT, the black holes do have a temperature. The temperature for any black hole can be given as:

$$T = \frac{\hbar\kappa}{2\pi} \tag{3.7.3}$$

Since we couple a QFT to a classical theory, the matter(quantum fields) outside the horizon will also contribute to the black hole entropy. Hence we define the generalised entropy of the black hole as sum of two quantities, first term contains the area of the horizon and second contains the entropy due to the outside-matter:

$$S_{gen} = \frac{\text{Area of the horizon}}{4\pi G_N} + S_{out} \tag{3.7.4}$$

where S_{out}:= Entropy of matter outside the horizon.
The generalised entropy is then found to obey the second law of thermodynamics i,e, $\Delta S_{gen} \geq 0$. Note that the generalised entropy includes the quantum effects and hence the contribution of Hawking radiation. The precise states giving rise to generalised entropy poses a much interesting problem which we won't be dealing with in this thesis.

3.7.2 Hawking Process

To analyse the problem of information paradox we must consider the case of black holes formed by gravitational collapse. These black holes will radiate like a thermal body and thus would evaporate. The analysis of eternal black holes showed that the radiation appears as entangled pair of particles. We interpret the Hawking process as pair creation of entangled particles near the horizon, one particle falls towards the singularity while the other escapes to infinity. The escaping particle appears to one as Hawking radiation. The black hole thus evaporates by emitting Hawking radiation. We now classify various stages of black hole evaporation.

(a) After gravitational collapse of stellar mass, a black hole forms and begins to evaporate. At early stages the outside of the black hole remains stationary. The geometry continues to elongate in one direction while the angular directions pinch towards zero size.

(b) The Hawking process now creates entangled pairs of particles. One of the particles escapes to infinity where it is observed as blackbody radiation.

(c) The black hole therefore begins to shrink as the mass is carried away by the radiation. Eventually the angular directions shrink to zero. The horizon also shrinks to zero. This is the singularity.

(d) Once the black hole reaches singularity it looses all its mass to radiation and there is a smooth spacetime containing thermal Hawking radiation but no black hole.

These stages of evaporation roughly describe the Hawking process. This process plays a vital role while formulating Hawking information paradox.

3.8 Salient Features of Quantum system for an evaporating black hole

The results we reviewed so far suggest that black hole can be regarded as an ordinary thermodynamic body obeying laws of thermodynamics. The Hawking process makes it necessary to analyse black hole using quantum field theory. Hence we regard black hole as a quantum system obeying laws of thermodynamics. Such a system can be categorised by finite but large number of degrees of freedom that obey ordinary laws of physics. To highlight various features of this quantum system in one statement, we refer to "central hypothesis":
As seen from the outside, a black hole can be described in terms of a quantum system with $(Area)/4G_N$ number of degrees of freedom. Such a quantum system would evolve unitarily under time-evolution.
We now state some important features of this quantum system comprising black hole:

- The hypothesis just mentions the black hole as seen from the outside. It does not make any remark on the black hole interior.
- The hypothesis makes no distinction between qubits, Fermions etc. describing the degrees of freedom. It just claims that the Hilbert space has finite dimension.
- The degrees of freedom mentioned in the statement are not manifest in the gravity description of the theory.
- Unitary evolution clearly implies that there exists a Hamiltonian that generates time evolution. However, this Hamiltonian is not manifest in the gravity description of the theory.
- In an asymptotically flat geometry, it is convenient to draw an imaginary surface surrounding the black hole and define the quantum system inside it. This imaginary surface is considered to be a few Schawrzschild radii away from the horizon. The quantum system is now coupled to

degrees of freedom living outside this surface. The region outside the imaginary surface is again a quantum system in fixed spacetime with no large fluctuations in the background. The full coupled evolution then must be unitary.

- Hence unitary evolution is possible if we surround the black hole by a reflecting wall and we should consider the full system inside this wall.
- The above statements suggest that gravity answers are compared to those of a quantum system that is coupled to degrees of freedom far from the black hole at imaginary cut-off surface.

The results from black hole thermodynamics as well as regarding the analysis of black hole entropy are a true property of theory of gravity coupled to quantum fields. Hence it becomes quite important to critically understand the quantum system of black hole.
In this thesis, instead of presuming that such a quantum system exists, we will prove the salient features of this quantum system as we try to understand the information paradox.

3.9 Entropy of Black hole: Fine-Grained or Coarse-Grained

In upcoming sections it will be clear that, the information paradox arises due to two different notions of entropy of any system. It is therefore useful to make some remarks on these two different notions of entropy.

3.9.1 Fine-Grained entropy

Given the density matrix ρ for any quantum state, von Neuman entropy is defined as:

$$S_{vN} = -Tr[\rho \log \rho] \tag{3.9.1}$$

Von Neuman entropy is often referred to as the fine-grained entropy of the system. For any pure state $S_{vN} = 0$, indicating complete knowledge of the quantum state. It is worth noting that S_{vN} is invariant unitary time evolution.

3.9.2 Coarse-Grained Entropy

There is yet another way in which entropy can be defined for any system. This is known as the coarse-grained entropy. We again have a density matrix ρ describing quantum state of the system but we do not measure all observables. We instead measure only a subset of coarse-grained observables say A_i. To calculate the coarse-grained entropy we follow the given procedure:

- Consider all possible density matrices $\tilde{\rho}$ which give the same result as our system for the tracked observables i.e. $Tr[\tilde{\rho}A_i] = Tr[\rho A_i]$,
- Further we compute the von Neuman entropy $\tilde{S(\rho)}$.
- Finally we maximise this for all possible choices of $\tilde{\rho}$.

A simple example for coarse-grained entropy is the thermodynamic entropy. Hence coarse-grained entropy obeys the second law of thermodynamics, it tends to increase under unitary time evolution.

3.9.3 Fine-grained vs. Coarse-grained

A few remarks are in order for the above mentioned notions of entropy.

- Consider a quantum system having two parts A and B, then the Hilbert space of the whole system can be given by $H = H_A X H_B$. We compute $S_{vN}(A)$ by forming density matrix ρ_A by taking partial trace over system B.
  Now note the fact that entropy of A can be non-zero since the whole system is in pure state. This indicates that A and B are entangled to each other. i.e. $S(A) = S(B) \neq 0$ but $S(A \cup B) = 0$.
- The fine-grained entropy can never be greater than coarse-grained entropy i.e. $S_{vN} \leq S_{coarse}$. This is due to the way we define the entropies since ρ can always be a candidate $\tilde{\rho}$.
- S_{coarse} provides a measure of total number of degrees of freedom available to the system. It serves as an upper bound to check the entanglement in the system.

3.9.4 Entropy for Black hole as a quantum system

In previous section we defined the generalised entropy of the black hole in equation (3.7.4). We know that S_{gen} follows second law of thermodynamics. Hence it is not invariant under unitary time evolution. The black hole entropy must therefore be the coarse-grained entropy of the system. Hence it will always tend to increase depending on the area of the horizon.
We can define fine-grained entropy of quantum fields in a region of space by considering slices of region. If Σ is a spatial region defined on some fixed time slice, we can always compute the associated density matrix ρ_Σ. We can then calculate $S_{vN}(\Sigma) = S_{vN}(\rho_\Sigma)$ as the fine-grained entropy for that particular spatial region Σ.
Now consider the quantum system of black hole. To define fine-grained entropy we consider a semi-classical approximation. A theory of classical geometry of spacetime is coupled to quantum fields defined on that particular geometry. We can then calculate $S_{semi-classical}(\Sigma)$. The fine-grained entropy therefore serves as the entropy over ρ_Σ computed by QFT in curved spacetime.
To summarise the generalised entropy for the quantum system of black hole serves as the coarse-grained entropy. On the other we can compute fine-grained entropy using semi-classical approximation of coupling quantum fields to a theory in classical relativity.

3.10 Information Loss Paradox

The Hawking information paradox is an argument that goes against the central hypothesis, mentioned in previous section. This happens when we assume that the hypothesis is true. To understand the paradox, we must first understand the origin of Hawking radiation.

3.10.1 Hawking radiation: A Qualitative Origin

The thermal aspects of Hawking radiation arise because we split the original vacuum state into two parts, the part in the black hole interior and the other part in the exterior. The vacuum state in QFT is an entangled state. If we consider the whole vacuum state, it is pure. However if we consider half of the space, we will surely obtain a mixed state on this half. This is a consequence of unitarity and relativistic invariance.

We can explain the above mentioned process in the following way. The vacuum state contains pairs of particles which are constantly created and annihilated. In presence of the horizon, one of the member can go to infinity while the other is trapped in black hole interior. We call the outward moving particle as "outgoing Hawking quantum" while the trapped particle is "interior Hawking quantum". These two particles are entangled to each other forming a pure state. However if we consider the outgoing Hawking quantum, it appears to us in a mixed state actually being a thermal state at Hawking temperature.
As a first guess the above mentioned process does not violate the central hypothesis. However, we need to closely examine the entropies arising due to this process.

3.10.2 The Paradox!!

We denote S_{rad} as the entropy of the emitted radiation, while S_{B-H} as the Bekenstein-Hawking thermodynamic entropy of the black hole. Consider the below mentioned remarks:

- In the early stages, the von Neumann entropy of emitted radiation would be same as that of the thermal entropy since the radiation is entangled with the quantum system.
- At early stages of evaporation therefore, S_{rad} increases.
- As black hole evaporates more and more, its area will shrink and we would arrive at a situation where entropy of radiation would become greater than the thermodynamic entropy of the black hole, i.e. $S_{rad} > S_{B-H}$.
- Once this happens, it is not possible for the entropy of the radiation to be entangled with quantum system describing the black hole because number of degrees of freedom of black hole is given by the thermodynamic entropy and hence the area of the horizon.
- If the black hole degrees of freedom together with radiation are forming a pure state, then the fine-grained entropy of the black hole i.e. S_{BH} must be equal to that of the radiation i.e. S_{rad} or $S_{BH} = Srad.$

- On the other hand the fine-grained entropy of the black hole must be always less than the coarse-grained one. i.e. $S_{BH} \leq S_{B-H}$.

The above remarks suggest that as time evolves unitarily, the entropy of radiation monotonically increases. On the other hand the thermodynamic entropy of the black hole decreases. However at some time, say t_P the value of both entropies becomes same. As time passes further the entropy of the radiation is more than the thermodynamic entropy of the black hole. But coarse-grained entropy can never be less than fine-grained entropy. This is the main essence of Hawking information paradox.

3.10.3 Page curve

At time t_P the entropy of radiation and thermodynamic entropy of the radiation become equal. If we expect the central hypothesis were true then unitary evolution of time implies that the entropy of the radiation must decrease for the time $t > t_P$. The curve thus obtained represents the unitary evolution of quantum system of black hole. The curve is known as Page curve and $t_P = t_{Page}$.
We express the paradox as in Figure 3.7. The red curve represents the thermodynamic entropy while the blue curve represents Hawking's results. The green curve on the other hand denotes the Page curve.

Hence the Hawking information paradox can be resolved if we obtain the Page curve since it respects the unitary evolution. Note that we need to obtain the Page curve separately for both entropy of radiation and entropy of black hole.

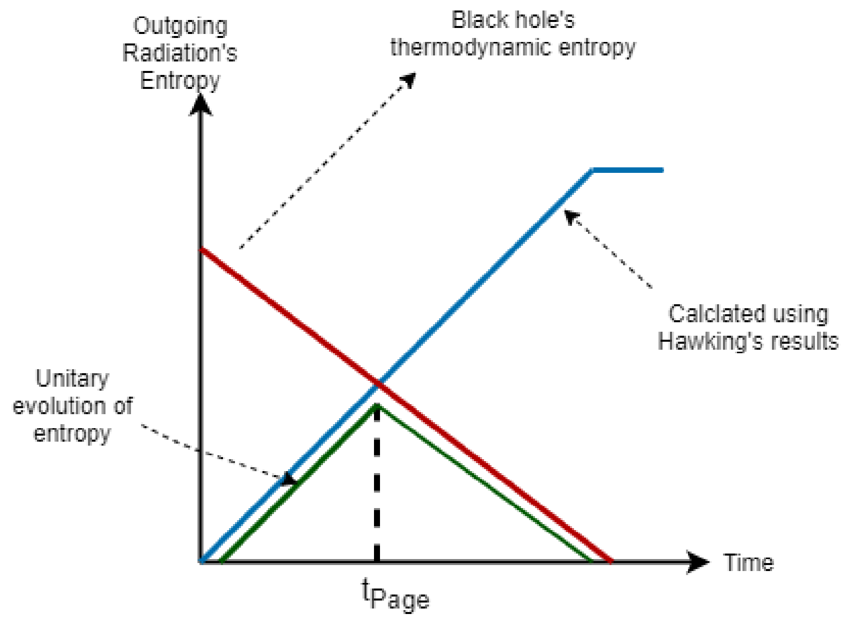

Figure 3.7: The information paradox and the Page curve

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3.11 Fine-Grained entropy in Gravitational systems

In previous section we attributed the coarse-grained entropy of black hole as the Bekestein-Hawking entropy. We now aim to find the fine-grained entropy of the black hole. Similar to the case of the B-H entropy, von Neumann entropy of the black hole can be given by a gravitational formula. The idea is that we choose a surface such that generalised entropy is minimised. The fine-grained entropy is then given as:

$$S \sim min\left[\frac{Area}{4G_N} + S_{outside}\right] \tag{3.11.1}$$

The precise formula is a bit different than the above mentioned equation (3.11.1). The surface we are considering is localised along one of the spatial dimensions and also in time. We aim to spatially minimise the entropy; but while evolving the system along time we need to maximise the temporal dependence. We therefore need to consider "extremal surfaces". If there are many extremal surfaces we must find the global minimum. We therefore choose a spatial slice and find the minimal surface. Then we find the maximum among all such choices of spatial slices. Hence the precise formula for fine-grained entropy becomes[5]:

$$S = min_X \left\{ \mathrm{ext}_X \left[\frac{Area(X)}{4G_N} + S_{\text{semi-cl}}(\Sigma_X) \right] \right\} \tag{3.11.2}$$

Here X is a codimension-2 surface, Σ_X is the region bounded by X and the cut-off surface, and $S_{\text{semi-cl}}(\Sigma_X)$ is the von Neumann entropy of the quantum fields on Σ_X appearing in the semi-classical description of the gravitational system.

To find such extremal surfaces we start from a surface outside the black hole and we may move it along the interior of black hole to find the minimum. The answer therefore depends on the geometry of the interior of the black hole. These extremal surfaces are often referred to as Quantum Extremal Surfaces(QES).

We can write the generalised entropy of the surface X as the quantity in brackets of equation (3.11.2);

$$S_{gen}(X) = \frac{Area(X)}{4G_N} + S_{\text{semi-cl}}(\Sigma_X) \tag{3.11.3}$$

While extremising we may take the surface to zero, the entropy then includes entropy of fields outside as well as that of the fields in the black hole interior. The entropy at such surfaces will therefore be equal to that of the star itself and not just the area of the horizon. Thus, fine-grained entropy does not change and obeys unitarity.

3.12 Entropy of an evaporating black hole

Considering the formula given by equation (3.11.2) we aim to obtain Page curve for entropy of black hole, by examining various stages of black hole evaporation. We employ the use of quantum extremal surfaces(QES) in our analysis.

3.12.1 Case I: Vanishing Extremal surfaces

We now compute the entropy at the very early stages of black hole evaporation, when the black hole would have just formed but before any Hawking radiation could escape the horizon. In this case, there are no extremal surfaces to be moved inwards. Instead we deform the QES X such that it shirnks to zero size. This QES is known as "vanishing extremal surface". In equation (3.11.2) the area term would therefore have zero contribution. The dominating contribution to the entropy is due to the matter fields. If we assume that the collapsing shell was in a pure state, then the matter-field contribution would vanish and hence for this case the entropy of the black hole would be zero. Since there is no Hawking radiation the fine-grained entropy of the black hole will remain invariant under time evolution. The QES for above described stage of black hole evolution is shown by Figure **??**.
At some point in time the black hole would start evaporating and Hawking quanta would be emitted outside the horizon. We can approximate the use of vanishing QES at this early stage of emission of Hawking radiation. As black hole starts emitting Hawking modes outside the horizon, the pure states get converted to mixed state. The dominating term would still be the entropy of matter fields, but the mixed states will tend to increase this entropy. Hence the generalised entropy given by equation (3.11.3) will tend to increase. Hence if we consider the case of vanishing extremal surfaces we would get an increasing trend of S_{gen}, the figure is given by Figure 3.8 [2].

3.12.2 Case II: Non-Vanishing extremal surfaces

As more Hawking quanta get emitted outside the horizon, we must consider non-vanishing QES. These surfaces lie close to the black hole event horizon. The QES can be constructed in the following way; We start off along the cutoff surface by a time of order of $r_s \log S_{BH}$. We now imagine shooting an ingoing light ray. Then the QES is located at the point where the light

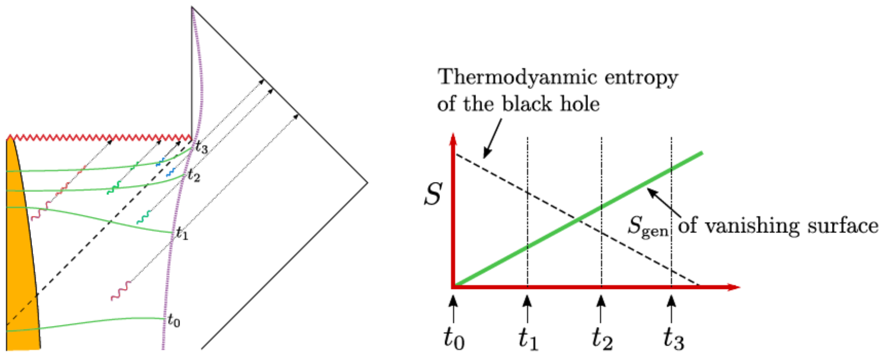

Figure 3.8: To the left: Construction of vanishing QES when Hawking modes escape horizon; To the right: S_{gen} due to vanishing QES.

ray intersects the horizon. The construction of such non-vanishing QES is represented by Figure 3.9 [2].
The generalised entropy seems to have both area-term and matter-term. The QES however does not capture enough Hawking modes and we may neglect the matter field term's contribution to the S_{gen}. Hence we approximate the entropy for this case as:

$$S_{gen}(X) \approx \frac{Area(X)}{4G_N} \tag{3.12.1}$$

Now as the black hole evaporates, the area decreases and hence we would have a decreasing trend in S_{gen}. The entropy due to non-vanishing QES therefore behaves in similar way to that of thermodynamic entropy and is represented by Figure 3.9.

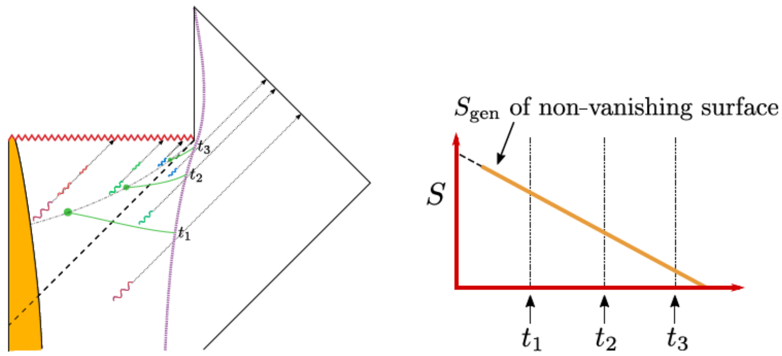

Figure 3.9: To the left: Construction of non-vanishing QES; To the right: S_{gen} due to non-vanishing QES.

3.12.3 Page curve for evaporating black hole

To obtain Page curve for the entropy of black hole, we need to take the minimum of S_{gen} over all the available choices of QES. We have two such choices; Vanishing QES and non-vanishing QES.

At very early times, only the vanishing surface exists giving a contribution to the entropy which starts at zero and monotonically increases until the black hole evaporates away.

Some short time after the black hole forms, the non-vanishing QES can be considered. The entropy now starts at large value and decreases as black hole shrinks.

Therefore S_{gen}(vanishing QES) initially captures the true fine-grained entropy of the black hole. As time passes the non-vanishing surface contribution becomes smaller and at some point in time, approximately at Page time, starts to represent the true fine-grained entropy of the black hole. Hence by transitioning between these two extremal surfaces, the black hole closely follows Page curve indicating unitary black hole evaporation.

The QES transitioning and the Page curve is represented by Figure 3.10 [2]. Here uptil t_1, the vanishing QES represents the entropy, however between t_1 and t_2 the surface transitions to non-vanishing QES. We then have a decreasing entropy till t_3. In this way we obtain Page curve for an evaporating black hole using extremal surfaces.

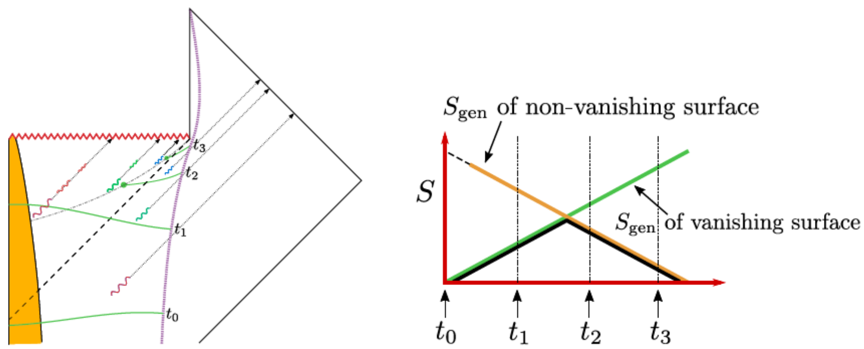

Figure 3.10: Construction of transitioning QES and the corresponding Page curve for entropy of black hole.

3.13 Entropy of Radiation

In section 3.12 we used semi-classical gravitational formula to obtain Page curve for black hole. The Hawking information paradox however concerns with that of obtaining Page curve for radiation. As a first intuition we claim that, the semi-classical gravitational formula can be used yet again to obtain Page curve for radiation.
The radiation resides in a spacetime region where the gravitational effects can be made very small since we approximate the region as rigid space. But, we used gravity to obtain this state, we can hence make an attempt to apply gravitational formula to compute entropy of radiation. Note that we are trying to apply the formula outside the cutoff region where there is no black hole. This is still valid because the formula allows us to move X in the interior so as to minimise the entropy.
We now consider the possibility that Σ_X can be disconnected. By doing so, we have increased the area of the boundary. We are now required to decrease the semiclassical entropy contribution. This can be achieved if we consider regions with entangled matter that are far away. The radiation is therefore entangled with black hole interior. We can hence decrease the semiclassical entropy contribution by including the black hole interior. Such disconnected QES are termed as islands.
We now modify the gravitational formula by including contribution of island term in area as well as semi-classical entropy. Using the fine-grained gravitational formula we compute the fine-grained entropy of the radiation

as:

$$S_{Rad} = min_X \left\{ \text{ext}_X \left[\frac{Area(X)}{4G_N} + S_{\text{semi-cl}}[\Sigma_{Rad} \cup \Sigma_{Island}] \right] \right\} \tag{3.13.1}$$

Now the area here corresponds to the area of the boundary of the island and we minimise and extremise with respect to the shape and location of island. The $S_{\text{semi-cl}}[\Sigma_{Rad} \cup \Sigma_{Island}]$ is the von Neumann entropy of the quantum state of the combined radiation and island systems in the semiclassical description. The formula in equation (3.13.1) essentially computes the entropy of the full quantum state of the radiation. Note that formula does not claim to extract this particular state in an explicit form.
We can therefore compute the generalised entropy as:

$$S_{gen} = \frac{Area(X)}{4G_N} + S_{\text{semi-cl}}[\Sigma_{Rad} \cup \Sigma_{Island}] \tag{3.13.2}$$

By using the formula we aim to compute the entropy of the Hawking radiation that has escaped from the black hole region. This is done by computing the entropy of the whole region from the cutoff surface to infinity. This is region is what Σ_{Rad} refers to in the formula. We can form any number of regions contained in the black hole side of the cutoff surface and term them as islands. We can consider any number of islands(including zero). If we have more than one extremal surfaces we again minimise the entropy over all such possible choices.

3.13.1 Case I: No Island Contribution

At earlier times we have no island. The vanishing island contribution lets us equate areal term to be zero and hence we have the entropy of the radiation as:

$$S_{gen} \approx S_{\text{semi-cl}}(\Sigma_{Rad}) \tag{3.13.3}$$

In this case we have more and more outgoing Hawking quanta which escape the black hole region and hence the entropy continues to grow. Referring to Figure 3.11 [2] it is clear that from t_0 to t_3 all the QES end up at the cutoff region and hence there are no islands to be considered. The corresponding variation in entropy is again represented in the same figure. Here the case of no island gives an ever increasing contribution to the entropy of radiation.

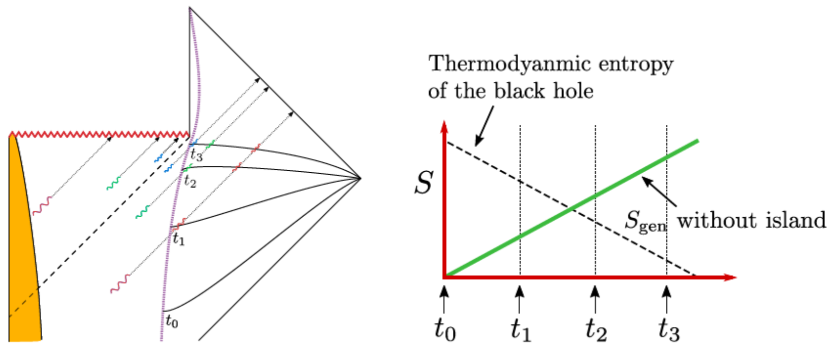

Figure 3.11: To the left: Construction of no island contribution; To the right: Corresponding variation in S_{gen} of radiation

3.13.2 Case II: Non-vanishing island contribution

At some later time, non-vanishing island that extremises the generalised entropy can be constructed. It can be shown that a time of order of scrambling time i.e. $r_s \log S_{BH}$ is enough to consider such non-vanishing islands. The island is centred around the origin and the boundary is near to the horizon. It can be moved up the horizon at different times on the cutoff surface. Therefore we need to consider the formula with the island contribution term i.e. equation (3.13.2). The island contains most of the interior Hawking modes that purify the outgoing radiation. The semi-classical entropy term contains the union of these modes and therefore can be neglected since all the modes of radiation will be purified by the island modes. Hence the S_{Rad} can be approximated as:

$$S_{gen} \approx \frac{Area(X)}{4G_N} \tag{3.13.4}$$

The area decreases as the black hole evaporates away, giving a decreasing contribution for the case of non-vanishing islands.
Figure 3.12 [2] shows the non-vanishing islands formed at t_1, t_2 and t_3 since disconnected regions appear at the other side of the horizon. The corresponding entropy contribution is also represented in the same figure.

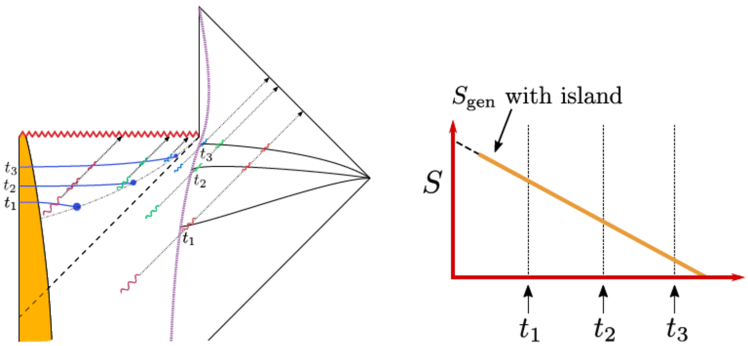

Figure 3.12: To the left: Construction of non-vanishing islands; To the right: Corresponding variation in S_{gen} of radiation

3.13.3 Page curve for Radiation

To obtain Page curve we need to minimise over all possible choices of island contribution. We have two cases:

- No island contribution gives an increasing contribution to the generalised entropy.
- The non-vanishing piece implies a decreasing contribution to the generalised entropy.

At early stages the no-island contribution turns out to be the minimised choice and hence we have a monotonically increasing entropy as time evolves. However at some later time approximately near the Page time, the minimised entropy turns out to be given by the non-vanishing island case. The entropy therefore switches its variation and starts decreasing. In this way the generalised entropy closely follows Page curve.
The corresponding transitioning between the two contributions is shown by Figure 3.13 [2]. From t_0 to t_1 we need to consider the no-island contribution and hence the generalised entropy starts increasing. Somewhere between t_1 and t_2 the entropy transitions due to appearance of islands. The entropy then decreases from this time to t_3 given by the non-vanishing island contribution. We therefore obtain the corresponding Page curve shown in Figure 3.13 [2].

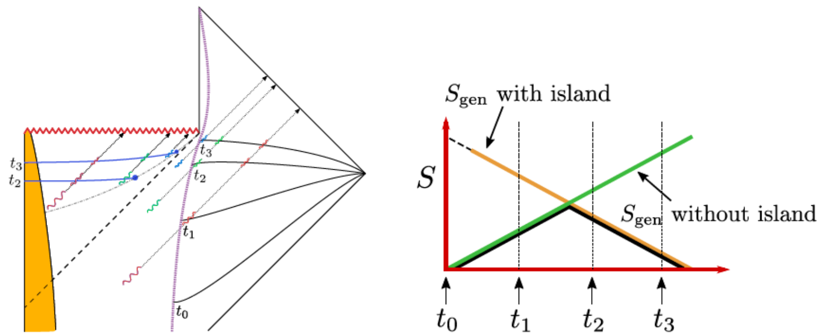

Figure 3.13: To the left: Transition between no-islands and non-vanishing islands; To the right: Corresponding Page curve for S_{gen} of radiation

3.14 Entanglement Wedge and Bags of Gold

In central hypothesis. the degrees of freedom which suffice to describe the black hole from outside are held of prime importance. The question however remains, whether these degrees of freedom describe the black hole interior. The formula for gravitational fine-grained entropy can be trusted to answer this question. The formula employs the use of geometry by using extremal surfaces and islands. We need to combine these two notions of minimal surfaces to describe these degrees of freedom. It is quite clear that the gravitational formula depends on the interior. Also, the formula depends on the state of quantum fields up to the extremal surface. We can hence assume that the degrees of freedom in the central hypothesis describe the geometry up to the minimal surface.

We define entanglement wedge as the surface area swept by the quantum extremal surface. Hence the entanglement wedge includes region from cutoff surface upto outside of QES. We claim that the black hole degrees of freedom are contained inside this entanglement wedge. Now we know that islands do describe black hole interior, hence black hole degrees of freedom might reside in the black hole interior, to some extent. We shall now describe entanglement wedge for various stages of black hole evaporation. It is to be noted that the black hole entanglement wedge shall describe black hole degrees of freedom, while the radiation entanglement wedge describes radiation degrees of freedom.

3.14.1 Case I: $t < t_{Page}$

For ealier stages the extremal surfaces turn out to be vanishing, hence these surfaces exist on $r = 0$. The corresponding entanglement wedge for black hole is then constructed from cutoff surface to the edge $r = 0$. This is denoted in Figure 3.14 [2] by the green coloured wedge.
At the same time the islands describing entropy of radiation do not exist at earlier stages. Hence we construct the entanglement wedge denoted by blue in Figure 3.14 [2].
It is clear that the black hole interior is contained by the entanglement wedge of the black hole. Hence the interior contains black hole degrees of freedom.

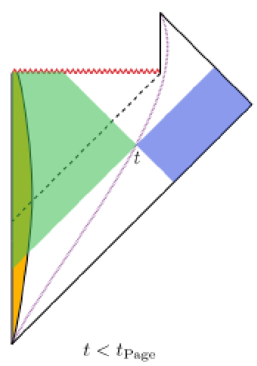

Figure 3.14: The entanglement wedge construction for $t < t_{Page}$

3.14.2 Case II: $t > t_{Page}$

Let us now examine the case for late times, larger than the Page time. The QES now exists in the interior of the black hole, near to the horizon. Also there is now a disconnected island describing the radiation. While constructing the wedges it is to be noted that only a part of interior of the black hole can be described by black hole degrees of freedom.
We then construct the radiation entanglement wedge shown in Figure 3.15 [2] by green. The wedge has disconnected regions as expected, due to non-vanishing islands. On the other hand we construct the black hole entanglement wedge as shown in Figure 3.15 [2] by blue. We construct the wedges in such a manner that the two never have common region.

It is then quite clear that for late times, most of the interior of black hole is described by radiation degrees of freedom. There exists a small portion of the interior containing black hole degrees of freedom.

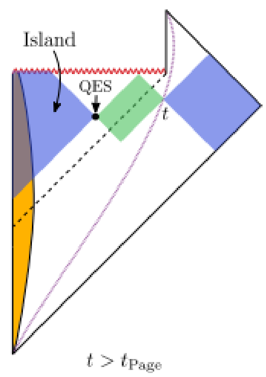

Figure 3.15: The entanglement wedge construction for $t > t_{Page}$

3.14.3 Case III: $t > t_{Evap}$

For very late times, when the black hole has completely evaporated, the region inside the cutoff surface is just flat space. The entanglement wedge of radiation will now include whole of the black hole interior. This is shown in Figure 3.16 [2] by green region. There exists a small region near to the cutoff surface belonging to black hole entanglement wedge, denoted by blue in Figure 3.16 [2].

3.14.4 The case of "Bags of Gold"

We now test the theory of entanglement wedge by attempting to resolve a paradox. It can be shown that there exist certain classical geometries which look like a black hole from the outside but the interior can contain arbitrarily very large entropy, larger than the area of the horizon. These surfaces are named as "bags of gold". The geometry has a narrow neck with a big "bag" containing some matter. For an outside observer, the geometry evolves into a black hole with area equal to that of the neck.

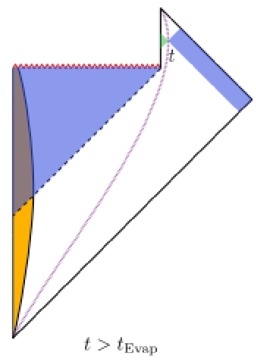

Figure 3.16: The entanglement wedge construction for $t > t_{Evap}$

3.14.5 Case I: Small entropy in the bag

Let us examine the case when the bag has little matter inside it and hence the entropy is small compared to the area of the horizon. We construct the minimal surface at the centre with zero area. Then the entanglement wedge will cover the whole bag. The bag contains less degrees of freedom, hence the wedge can be constructed in the above prescribed manner. The entropy is then clearly lesser than the area of the neck(horizon).

3.14.6 Case I: Large entropy in the bag

We construct the minimal surface such that it is of the order of the area of the horizon. The minimal surface is then constructed on the neck. The wedge thus constructed, does not cover the bag. The bag is allowed to contain a large entropy since this entropy is not described by black hole degrees of freedom.
We have therefore resolved the paradox using entanglement wedge construction. The geometries can then exist such that the interior contains larger entropy than the area of the horizon.

3.15 Concluding Remarks

We now conclude the analysis of quantum mechanical treatment of black holes and mention some important remarks about the same.

- By applying the AdS/CFT correspondence to the case of eternal black hole, we could form the Hartle-Hawking state.
- The H-H state helped to understand the Hawking process.
- The analysis of eternal black holes asserts the fact that Hartle-Hawking state is an entangled state and hence quantum entanglement must play an important role in the analysis of information paradox.
- If we formed black hole from an initially pure state, we expect that entropy of black hole region and that of radiation region must be equal.
- By considering QES we can reproduce Page curve for black hole entropy.
- By considering extremal islands we can reproduce Page curve for entropy of radiation.
- The entanglement wedge construction is used to describe the degrees of freedom mentioned in central hypothesis.
- At very early times the black hole interior can be described by black hole degrees of freedom.
- At late times the black hole interior has radiation degrees of freedom in most part.
- Once the black hole has completely evaporated the whole interior is described by radiation degrees of freedom.

Chapter 4

String Theory

4.1 A first look at Strings

In the upcoming sections we investigate the study of the last model for this thesis namely the bosonic string theory. String theory can be considered as a healthy candidate for a theory of everything. In this section we attempt to give a brief introduction of string theory by focusing on the most important question "Why do we need strings to describe such a theory?"

4.1.1 Quest for a quantum theory of gravity

The current understanding of theoretical physics elucidates that all the interactions in the universe are caused by the fundamental forces viz., gravitational, electromagnetic, weak nuclear force and strong nuclear force. One of the main themes of research in theoretical physics has been to develop a model which can unite all these forces. In recent years it has been established that the principles underlying the understanding of nature must obey the rules of quantum mechanics. The unification therefore must be consistent within the quantum mechanical framework.

It is well established that quantum field theory unifies the electromagnetic force, weak and strong nuclear forces in form of gauge theories as $SU(3)XSU(2)XU(1)$ model or Standard Model. Standard Model when combined with general relativity is consistent with all experimental evidences upto Planck scale.

The Standard model however seems to be incomplete. It does not explain the reasons behind, choosing only a particular pattern for gauge fields and different parameters in Lagrangian while unification. Secondly, the unification of quantum field theory with general relativity leaves us with a non-renormalisable theory. On the other hand even at the classical level the theory breaks down at singularities of general relativity. Although the experimental evidences reaffirm the status of Standard model we must find a guide to a more complete theory. One therefore seeks to unify standard model and gravity in a simpler structure as a quantum theory of gravity. Such a theory must be consistent with ideas of grand unification, extra space-time dimensions and super-symmetry.

4.1.2 Why Strings?

At Planck scale(high energies) the theory of quantum gravity must not give divergences. Presently we have only one way to spread out gravitational interaction and cut off the divergence without breaking the symmetries of the theory.This is the string theory. In this theory all the elementary particles are considered as one-dimensional objects, strings, rather than points as in quantum field theory. Since we have at least one solution to cure out the divergences, it is quite important to investigate string theory.
Further string theory has connections to many areas of mathematics and has led to many new ideas in these areas. String theory shares rich bonds with supersymmetric quantum field theories. String theory also addresses some of the deeper questions of quantum mechanics of black holes.
Whether we are trying to cut off the divergences of quantum gravity, or we are exploring the shortcomings of Standard model or searching for new symmetries or new mathematical formulations, we are led to string theory.
In the thesis we do not aim to study string theory to a great depth, rather we take the classical theory of simplest strings and attempt to quantise the same. The theory thus formed is termed as Bosonic strong theory.

4.2 Action principles for Bosonic strings

We aim to find the action governing the dynamics of bosonic strings. The action thus formed will help us to analyse the guage freedom enjoyed by the bosonic strings. In turn, the classical dynamics of string will thus help us to quantise it. We employ the standard convention; using Greek indices (e.g. $\nu = 0, 1, 2...$) to label spacetime coordinates and Latin indices (e.g. $i = 1, 2, 3...$) to label spatial coordinates. Throughout the analysis we consider a d dimensional theory with the signature of Minkowski metric as:
$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1... + 1)$.

4.2.1 Dynamics of a Relativistic particle

A bosonic string can be considered as a one dimensional extension of point particle. It seems promising to tackle this problem by revisiting the case of a relativistic point particle.

In a relativistic theory, the action of a point particle of mass m, is given by:

$$S = -m \int dt \sqrt{1 - \dot{X}_i \dot{X}^i} \tag{4.2.1}$$

Here X represent the spacetime coordinate of the background. As a first glimpse, it appears that temporal coordinate acts as a label while the spatial coordinates are dynamical degrees of freedom. The action therefore does not seem to be Lorrentz invariant.
There are two ways to make the above action Lorrentz invariant, the first one being, treating both space and time as labels. However this will make our theory to be treated using field heuristic approach.
The second way is to promote time to be a dynamical degree of freedom. Once we do this the system must move in time, it doesn't have a choice. We therefore treat time as fake degree of freedom using parametrisation. Consider the new action to be:

$$S = -m \int d\tau \sqrt{1 - \dot{X}_\mu \dot{X}^\mu} \tag{4.2.2}$$

Here τ is a new parameter such that,

$$\dot{X}_\mu := \frac{dX_\mu}{d\tau} \tag{4.2.3}$$

The action in equation (4.2.2) characterises the worldline of the particle and τ parametrises the motion along the worldline. It serves as the trajectory of the particle in spacetime. The worldline of a particle can be represented by Figure 4.1 where notches correspond to different values of τ. Also note that the action measures the length of this worldline.

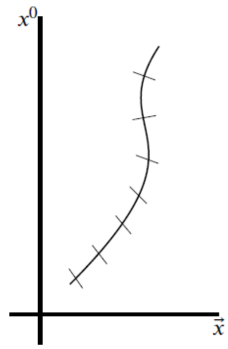

Figure 4.1: The Worldline for a point particle

The action in equation (4.2.2) is invariant under reparametarisation $\tau \to \tilde{\tau}(\tau)$. The reparametarisation invariance lets us define time as a new temporal coordinate while retaining the total number of degrees of freedom. The invariance therefore serves as the gauge symmetry for the particle action.

4.2.2 The Nambu-Goto Action

Let us try to evaluate the action of a relativistic string, analogous to that of the point particle. Since string is considered as one dimensional extension of point particle, the worldline of particle shall translate to worldsheet of the string. The worldsheet will be two-dimensional and hence we are required to parameterise this worldsheet using two labels. Finally we want the action of the string to measure area of this worldsheet.

Method I: Using worldsheet coordinates

We use the labels σ for space and τ for time to parametrise the worldsheet of the string. We need to assume the case of closed strings, $\sigma \in [0, 2\pi)$ with periodicity $\sigma(x) = \sigma(x + 2\pi)$. The worldsheet thus obtained can be represented by Figure 4.3.

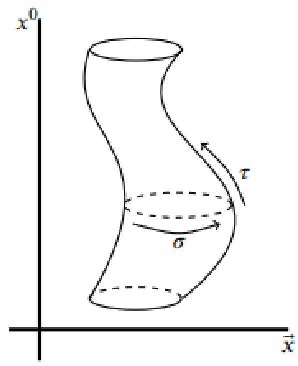

Figure 4.2: The Worldsheet for a closed string.

We denote the worldsheet coordinates by $\sigma^\alpha = (\sigma, \tau)$ where $\sigma^1 = \sigma$ and $\sigma^2 = \tau$. Note that we have two sets of coordinates now:

- $\{X^\mu\}$:= Background spacetime coordinates
- $\{\sigma^\alpha\}$:=Worldsheet coordinates

We further specify that; $\partial_1 X^\mu := \frac{\partial X^\mu}{\partial \sigma}$ and $\partial_2 X^\mu := \frac{\partial X^\mu}{\partial \tau}$.

Now consider a Euclidean 2-space. If $d\vec{l}_1$ and $d\vec{l}_2$ are two infinitesimal vectors for a tile such that θ denotes the angle between the vectors, then we calculate the area of the tile as:

$$dA = |d\vec{l}_1||d\vec{l}_2| \sin\theta \tag{4.2.4}$$

Consider a matrix M given by: $M = \begin{pmatrix} |d\vec{l}_1|^2 & |d\vec{l}_1|.|d\vec{l}_2| \\ |d\vec{l}_1|.|d\vec{l}_2| & |d\vec{l}_2|^2 \end{pmatrix}$

Then the area of the tile can be rewritten as:

$$dA = \det M \tag{4.2.5}$$

We can translate the above problem in Minkowski background spacetime by denoting:

$$d\vec{l}_1 = \frac{dx}{d\sigma} \qquad \textit{and} \qquad d\vec{l}_2 = \frac{dx}{d\tau} \tag{4.2.6}$$

The action then can be derived to be:

$$S = -A \int d\sigma d\tau \sqrt{\text{-det}(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{4.2.7}$$

Here the α and β indices denote the entries of the matrix M. We further denote the matrix entries as, $\dot{X}^\mu := \partial_1 X^\mu$ and $X^{\mu\prime} := \partial_2 X^\mu$.

Using these symbols the matrix can be rewritten as: $M = \begin{pmatrix} (\dot{X})^2 & \dot{X}.X' \\ \dot{X}.X' & (X')^2 \end{pmatrix}$

Then the determinant becomes:

$$\text{det}M = (\dot{X}^2)(X')^2 - (\dot{X}.X')^2 \tag{4.2.8}$$

The action then becomes:

$$S = -A \int d\sigma d\tau \sqrt{\dot{X}^2)(X')^2 - (\dot{X}.X')^2} \tag{4.2.9}$$

If we denote the symbol A as the tension, T (reasons mentioned in upcoming sections), we get the action as:

$$S = -T \int d\sigma d\tau \sqrt{\dot{X}^2)(X')^2 - (\dot{X}.X')^2} \tag{4.2.10}$$

This is known as the Nambu-Gotto action for string. Note that action is inherently become Lorrentz invariant. Also the worldsheet coordinates are both gauge symmetries and therefore convey that the action is invariant under reparameterisation.

Method II: Using induced metric

The action can be evaluated by yet another method, considering the worldsheet metric. The spacetime metric can be used to induce a metric over worldsheet. This induced metric can then be used to calculate area of worldsheet which in turn helps us to find the action. The worldsheet metric is then termed as induced metric and is given by the pullback of the flat Minkowski metric.

Consider a manifold $\mathcal{M}$ equipped with a metric g: $(\mathcal{M}, O, A, g)$ where O and A are topology and atlas respectively. Next, consider a submanifold

$\mathcal{N} \subset \mathcal{M}$. Let us associate the submanifold $\mathcal{N}$ with coordinates ξ^a where $a = 1, 2, ...\text{dim}N$. The coordinates then fetch us information about the embedding of $\mathcal{N}$ into $\mathcal{M}$. We embed $\mathcal{N}$ into $\mathcal{M}$ by defining functions $X^\mu(\xi^a)$; $\mu = 1, 2, ...\text{dim}M$. We can further induce a metric γ on $\mathcal{N}$ from g by using the following expression:

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{4.2.11}$$

In the case of induced metric for the worldsheet of string, the spacetime metric corresponds to $\eta_{\mu\nu}$ while the induced metric is $\gamma_{\alpha\beta}$. Then we can write:

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{4.2.12}$$

The action can then be derived as:

$$S = -T \int d\sigma d\tau \sqrt{-\det(\gamma)} \tag{4.2.13}$$

where we define the matrix γ as: $\begin{pmatrix} (\dot{X})^2 & \dot{X}.X' \\ \dot{X}.X' & (X')^2 \end{pmatrix}$. Equation (4.2.13) then resembles the Nambu-Gotto action.

Tension of the string

We now wish to understand attributing the constant A in Nambu-Gotto action as tension of the string. Gauge symmetry can be used to fix this constant. Let us choose a gauge such that the temporal coordinate $t := x^0 = R\tau$ for some dimensionful parameter R. Now consider a snapshot of the string such that its instantaneous kinetic energy vanishes. Thus, $\frac{\partial \vec{x}}{\partial \tau} = 0$. Then the matrix M in Nambu-Gotto action becomes: $\begin{pmatrix} (d\vec{x}/d\sigma)^2 & 0 \\ 0 & R^2 \end{pmatrix}$
The action can be evaluated as:

$$S = -T \int d\sigma d\tau R \sqrt{(d\vec{x}/d\sigma)^2} = -T \int dt L \tag{4.2.14}$$

where L is the spatial length of the string. We know that the action has dimensions of energy, hence the constant T must have dimensions of energy per unit length, which attributes to that of tension. Hence T can be considered as the tension of the string. It can be shown that this tension is:

$$T = \frac{1}{2\pi\alpha'} \tag{4.2.15}$$

where α' appears as a dimensionless constant in the equation for Regge trajectory. Hence finally the Nambu Gotto action becomes:

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-(\dot{X})^2(X')^2 - (\dot{X}.X')^2} \tag{4.2.16}$$

Symmetries of Nambu-Gotto Action

Let us revisit the symmetries obeyed by the above discussion of Nambu-Gotto action.

- The action is inherently Poincare invariant and therefore doesnot depend on the worldsheet coordinates.
- The action is invariant under reparameterisation. This exhibits the gauge symmetry of the action.

4.2.3 Polyakov Action

The Nambu-Gotto action still seems to be a very complex and difficult one to quantise. We need to massage it into a neat and much simpler action by attempting to remove the square root. The action thus formed is easy to quantise. We define the Polyakov action as:

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{4.2.17}$$

where $g := \det g_{\alpha\beta}$. Note that g is not the induced metric here, we have not used the flat Minkowski metric to obtain g. Rather g is an independent variable. g can be considered as the dynamical metric of the worldsheet.
We define Weyl transformation as one associated with the local rescaling of metric. The metric when Weyl transformed becomes:

$$g \to e^{2\phi(x)}g \tag{4.2.18}$$

The transformation therefore maps a metric to another metric of the same conformal class.

Weyl invariance of the action

We now show that Polyakov action is Weyl invariant. The metric must be transformed to $g_{\alpha\beta} \to e^{2\phi(x)} g^{\alpha\beta}$. The action when Weyl transformed must lead to the trivial result. Let us now vary action with respect to $g^{\alpha\beta}$;

$$0 = \delta S = \delta g^{\alpha\beta}\left(\sqrt{-g}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\sqrt{-g} g^{\rho\gamma}\partial_\rho X^\mu \partial_\gamma X_\mu\right) \tag{4.2.19}$$

If we perform Weyl transformation on the above equation:

$$R.H.S. = e^{\phi(x)} g^{\alpha\beta}\left(\sqrt{-g}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\sqrt{-g} g^{\rho\gamma}\partial_\rho X^\mu \partial_\gamma X_\mu\right) \tag{4.2.20}$$

Hence we will obtain:

$$R.H.S. = e^{\phi(x)}\sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{}{2} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right) = 0 \tag{4.2.21}$$

Hence it is clear that the Polyakov action is Weyl invariant since $L.H.S. = R.H.S. = 0$.

Equivalence to Nambu-Gotto action

We claim that the Polyakov action is classically equivalent, upto a gauge, to the Nambu-Gotto action. From equation (4.2.19) one can surely find:

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta} g^{\rho\gamma}\partial_\rho X^\mu \partial_\gamma X_\mu \tag{4.2.22}$$

Taking the negative of determinant on both sides and then sqaure root, from the above equation we obtain:

$$\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} = \sqrt{-\det\left(\frac{1}{2} g_{\alpha\beta} g^{\rho\gamma}\partial_\rho X^\mu \partial_\gamma X_\mu\right)} \tag{4.2.23}$$

By using identities of determinant we get the following:

$$\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} = \sqrt{-g}\frac{1}{2} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{4.2.24}$$

Hence the two actions are equivalent to each other. Plugging either side of equation (4.2.24) into respective actions gives the other action.
However the two actions are equivalent only if we take the Weyl transformation of metric as the equivalence class. This eqivalence class then corresponds to a redundancy in the solutions and hence acts as a gauge symmetry.

Fixing Gauge

The Polyakov action enjoys three gauge symmetries namely:

1. Diffeomorphism in σ
2. Diffeomorphism in τ
3. Weyl invariance

The worldsheet metric also has three independent components. So, we can use these gauges to fix them and transform Polyakov action into a neat form. The gauge symmetries are first used to set $g_{\alpha\beta}$ such that it becomes a flat metric: $ds^2 = -d\tau^2 + d\sigma^2$. We can use diffeomorphisms to give us something that is locally conformal to the flat metric:

$$g_{\alpha\beta} = e^{\phi(\sigma)}\eta_{\alpha\beta} \tag{4.2.25}$$

Weyl invariance can then be used by setting $\phi(\sigma) = 0$ to obtain the flat metric. Using the flat metric the Polyakov action becomes:

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \partial_a X^\mu \partial^a X_\mu \tag{4.2.26}$$

Now we need to address the question, whether we have used all the gauge freedom in setting the metric to the flat one and then obtaining the above action. We can find diffeomorphisms that change the metric by a Weyl factor. If such is the case we should have some gauge freedom left.
Consider the change of coordinates to worldsheet light-cone coordinates:

$$\sigma_\pm := \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{4.2.27}$$

We can the show that:

$$2d\tau^2 = d\sigma_+^2 + d\sigma_-^2 + 2d\sigma_+ d\sigma_- \tag{4.2.28}$$

Similarly we can show that:

$$2d\sigma^2 = d\sigma_+^2 + d\sigma_-^2 - 2d\sigma_+ d\sigma_- \tag{4.2.29}$$

Hence the metric becomes:

$$ds^2 = -2d\sigma_+ d\sigma_- \tag{4.2.30}$$

If we now let $\sigma_+ = f(\tilde{\sigma_+})$ and $\sigma_- = g(\tilde{\sigma_-})$ as some functions then the metric transforms to :

$$ds^2 = f'(\tilde{\sigma_+})g'(\tilde{\sigma_-})d\tilde{\sigma_+}d\tilde{\sigma_-} \tag{4.2.31}$$

The product on the R.H.S. can be removed by using Weyl transformation. Therefore we can have a diffeormorphism using Weyl transformation.

We further note that using lightcone coordinates it can be shown that:

$$\partial_+ X \partial_- X = -\frac{1}{2}\partial_a X \partial^a X \tag{4.2.32}$$

Hence the Polyakov action in flat spacetime corresponds to:

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \partial_+ X^\mu \partial^- X_\mu \tag{4.2.33}$$

where $\partial_+ = \frac{\partial}{\partial\sigma_+}$ and $\partial_- = \frac{\partial}{\partial\sigma_-}$

Hence we have transformed the Nambu-Gotto action in the form of equation (4.2.33) using Weyl invariance of Polyakov action. The action thus formed is more precise and as we will see in upcoming sections, is easier to quantise.

4.3 Quantising the String: Part I

In previous section we attempted to fix the gauge symmetries using conformal diffeomorphisms and Weyl invariance. We arrived at a much simpler Polyakov action in the form of equation (4.2.33). The action can further be used to evaluate the equation of motion in the form of:

$$\partial_+ \partial^- X^\mu = 0 \tag{4.3.1}$$

The problem is similar to that of wave equation and the most general solution to the above equation can be stated as:

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-) \tag{4.3.2}$$

where $X_L^\mu(\sigma_+) :=$ Left moving wave and $X_L^\mu(\sigma_+) :=$ Right moving wave. We can expand the solution in various modes and then try to fix the remaining gauges.

4.3.1 Mode expansion of string

We expand equation (4.3.2) in Fourier modes as follows:

$$X_L^\mu(\sigma_+) = a^\mu + b^\mu \sigma_+ + \sum_{n \neq 0} c_n^\mu e^{-in\sigma_+} \tag{4.3.3}$$

$$X_R^\mu(\sigma_-) = \tilde{a}^\mu + \tilde{b}^\mu \sigma_- + \sum_{n \neq 0} \tilde{c}_n^\mu e^{-in\sigma_-} \tag{4.3.4}$$

Using lightcone coordinates we can write:

$$\sigma_+ + \sigma_- = 2\tau \tag{4.3.5}$$

We recall that we considered periodic strings i.e. $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$.

We demand that the modes follow the periodicity, thus $b^\mu = \tilde{b}^\mu$. This will give an overall period term in the total expansion of $X^\mu(\tau, \sigma)$ such that,

$$b^\mu(\sigma_+ + \sigma_-) = 2b^\mu \tau \tag{4.3.6}$$

This term is then clearly unaffected by the periodicity.

Let us define the action, independent in σ as:

$$S = \frac{1}{2\alpha'} \int d\tau (\dot{X}^+)^2 \tag{4.3.7}$$

The Lagrangian can be therefore defined as:

$$L = \frac{1}{2\alpha'}(\dot{X}^+)^2 \tag{4.3.8}$$

We can then evaluate conjugate momentum as:

$$p^+ = \frac{1}{\alpha'}\frac{dX^+}{d\tau} \tag{4.3.9}$$

Integrating above equation and using equation (4.3.6) we get:

$$X^+ = \frac{1}{2}\alpha' p^+(\sigma_+ + \sigma_-) \tag{4.3.10}$$

In terms of left and right moving wave modes we can split the above equation:

$$X_L^+ = \frac{\alpha' p^+}{2}\sigma^+ \tag{4.3.11}$$

$$X_R^+ = \frac{\alpha' p^+}{2}\sigma^- \tag{4.3.12}$$

Similar analysis can be done to evaluate X^- At first glance it would seem to one that X^μ has d degrees of freedom. However we must consider the string to oscillate only along transverse directions. The longitudinal degree of freedom remains inaccessible to the motion of string. Also it sounds absurd to talk about movement along time dimension. Hence we actually need to fix the $d-2$ degrees of freedom.
Using X^+ we fixed one of these degree of freedoms by constricting the system to only spatial dimensions. Hence we can split the coordinate as: $X^+ := t \pm X^{d-1}$.

We have therefore managed to translate our problems to that of fixing $d-1$ degrees of freedom. We can attempt to fix one more gauge by considering the following equation of motion for string:

$$\partial_+ X^\mu \partial_+ X_\mu = 0 \tag{4.3.13}$$

We can expand the above equation as:

$$0 = -\partial_+ t \partial_+ t + \sum_{i=1}^{d-1} \partial_+ X^i \partial_+ X_i$$

Since X_L^+ and X_L^- depend on σ_+ one can write the above equation in the form:

$$0 = -\partial_+ X_L^+ \partial_+ X_L^- + \sum_{j=1}^{d-2} \partial_+ X_L^j \partial_+ X_{j,L} \tag{4.3.14}$$

Using equation (4.3.11) we can therefore write:

$$\frac{\alpha' p^+}{2} \partial_+ X_L^- = \sum_{j=1}^{d-2} \partial_+ X_L^j \partial_+ X_{j,L} \tag{4.3.15}$$

Proceeding in similar way we can evaluate:

$$\frac{\alpha' p^+}{2} \partial_- X_R^- = \sum_{j=1}^{d-2} \partial_- X_R^j \partial_- X_{j,R} \tag{4.3.16}$$

Hence we can determine X^-, given that we know the $d-2$ scalar field solutions. We take note of the fact that we have reduced our problem to finding these $d-2$ transverse modes of string motion.

We can regard the constants a^μ and its counterpart in equations (4.3.3) and (4.3.4) as the zero modes of the string motion. We can then use equations (4.3.6),(4.3.11) and (4.3.16) to finally renormalise the mode expansion of the string motion. The final result would then turn out to be:

$$X_L^\mu(\sigma_+) = \frac{1}{2} X_0^\mu + \frac{1}{2} \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma_+} \tag{4.3.17}$$

$$X_R^\mu(\sigma_-) = \frac{1}{2} X_0^\mu + \frac{1}{2} \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma_-} \tag{4.3.18}$$

Note that the form of the mode expansion coefficients i.e. α_n^μ will be clear once we quantise the string. The final mode expansion equations defined above will help us in quantising the string.

4.3.2 Poisson brackets and Symplectic forms

A Poisson bracket is a bilinear operator defined as:

$$\{.,.\} : C^\infty(F) \quad X \quad C^\infty(F) \to C^\infty(F) \tag{4.3.19}$$

where $C^\infty(F)$ is the space of smooth functions on phase space F of the system. We can state some of the properties of Poisson brackets as:

- $\{f, g\} = -\{g, f\}$
- $\{fg, h\} = f\{g, h\} + \{f, h\}g$
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

We can always find some set of cannonical coordinates such that Poisson brackets take the form:

- $\{q^i, p^j\} = \delta_{ij}$
- $\{p^i, p^j\} = \{q^i, q^j\} = 0$

Let $\mathcal{M}$ be a $2n$-dimensional manifold. A two form ω on $\mathcal{M}$ that is globally non-degenerate is defined by:

$$\omega(X, Y) = 0 \quad \text{such that} \quad \forall\ Y, X = 0 \tag{4.3.20}$$

The 2-form ω is closed i.e. $d\omega = 0$. Then ω is said to be symplectic form. $\mathcal{M}$ is said to be a symplectic manifold. We claim that the phase space of a system is a symplectic manifold.
We can therefore find local coordinates $\{q^i, p^i\}$ for $i = 1, 2...n$ such that, the symplectic form becomes:

$$\omega = dq^i \Lambda dp_i \tag{4.3.21}$$

Using the above symplectic form we can redefine Poisson bracket for cannonical coordinates as:

$$\{f, g\} = \omega\{X_f, X_g\} = L_{X_g} f \tag{4.3.22}$$

where L implies taking the respective Lie derivative. Therefore $X_{p_i} = \frac{\partial}{\partial q^i}$ and $X_{q_i} = -\frac{\partial}{\partial p^i}$.

The symplectic form can be evaluated using yet another definition:

$$\omega = \frac{1}{2}\omega_{nm} de^n \Lambda de^m \tag{4.3.23}$$

where ω_{nm} is a $2d\ X\ 2d$ antisymmetric matrix such that; $\{e^n, e^m\} = \omega^{nm}$ is its inverse.

The Poisson bracket relations for a classical system can be quantised by promoting the cannonical coordinates to operators and hence finding the commutation relations:

$$[q^i, p_j] = i\hbar\{q^i, p_j\} \tag{4.3.24}$$

This technique of quantising a system is known as cannonical quantisation.

4.3.3 Cannonical Quantisation for String

Using the techniques mentioned in previous subsection, we can attempt to perform cannonical quantisation for the string. We need to first find the symplectic form by evaluating cannonical momentum. Once we get the anti-symmetric matrix by using symplectic form, we can invert the matrix. The 2-form and matrix can then be used to find the respective Poisson bracket relations. Finally we can quantise those brackets and evaluate commutation relations.

Using the Polyakov action given by equation (4.2.17) we can calculate the cannonical momentum of the string as:

$$p_\mu = \frac{\partial}{\partial \dot{X}^\mu}\left(\frac{1}{4\pi\alpha'}\dot{X}_\mu^2\right) \tag{4.3.25}$$

Therefore we obtain:

$$p_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu \tag{4.3.26}$$

Using the cannonical coordinates we write the symplectic form as:

$$\begin{aligned} \omega &= \frac{1}{2}\int d\sigma\ dX^\mu \Lambda\ dp_\mu \\ &= \frac{1}{4\pi\alpha'}\int d\sigma\ dX^\mu \Lambda\ d\dot{X}_\mu \end{aligned} \tag{4.3.27}$$

If we use the lightcone coordinates by defining $X^\pm$ to fix the gauge, we are left with $2d-1$ degrees of freedom. Thus the symplectic form becomes:

$$\omega = \frac{1}{4\pi\alpha'}\int d\sigma \left(-dX^+ \Lambda\ d\dot{X}^- - dX^- \Lambda\ d\dot{X}^+ + 2\sum_{i=1}^{2d-1} dX^i\, d\dot{X}^i\right) ?? \tag{4.3.28}$$

By definition we know that X^+ is a purely zero mode term, i.e. it does not contain any oscillator modes. On the other hand X^- contains oscillatory modes. On performing wedge products of zero modes and oscillatory modes, the corresponding integrals would vanish. Hence for evaluating symplectic form we need to consider the third term in equation (**??**). Hence we can write:

$$\omega = \frac{1}{2\pi\alpha'} \int d\sigma \left(\sum_{i=1}^{2d-1} dX^i \, d\dot{X}^i \right) \tag{4.3.29}$$

The mode expansion shown in equations (4.3.17) and (4.3.18) can be rewritten as:

$$X^i = X_0^i + \alpha' p^i \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^i}{n} e^{-in\sigma_+} + \frac{\tilde{\alpha}_n^i}{n} e^{-in\sigma_-} \right) \tag{4.3.30}$$

We first deal with the oscillatory modes; the derivatives of equation (4.3.30) can then be written as:

$$dX^i = i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{d\alpha_n^i}{n} e^{-in\sigma_+} + \frac{d\tilde{\alpha}_n^i}{n} e^{-in\sigma_-} \right) \tag{4.3.31}$$

$$d\dot{X}^i = \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \left(d\alpha_m^i e^{-im\sigma_+} + d\tilde{\alpha}_m^i e^{-im\sigma_-} \right) \tag{4.3.32}$$

Substituting equations 4.3.31 and 4.3.31 in equation 4.3.29 the symplectic form becomes:

$$\omega = \frac{i}{2} \sum_{i=1}^{d-2} \sum_{n,m \neq 0} \int d\sigma \left\{ \left(\frac{d\alpha_n^i}{n} e^{-in\sigma_+} + \frac{d\tilde{\alpha}_n^i}{n} e^{-in\sigma_-} \right) \Lambda \left(d\alpha_m^i e^{-im\sigma_+} + d\tilde{\alpha}_m^i e^{-im\sigma_-} \right) \right\} \tag{4.3.33}$$

We get four terms from the wedge products in above integral viz.:

- If we consider wedge product term $d\alpha_n \, \Lambda \, d\alpha_m$ we get exponential $e^{-i(n+m)\sigma}$, this integral therefore remains only if $m = -n$.
- The above argument applies for $d\tilde{\alpha}_n \, \Lambda \, d\tilde{\alpha}_m$ wedge product term.
- If we consider the two cross terms: $d\alpha_n \, \Lambda \, d\tilde{\alpha}_n + d\tilde{\alpha}_n \, \Lambda \, d\alpha_n$, the integral vanishes.

Hence we are left with following symplectic form:

$$\omega = \frac{i}{2} \sum_{i=1}^{d-2} \sum_{n \neq 0} \frac{1}{n} \left(d\alpha_n^i \wedge d\alpha_{-n}^i + d\tilde{\alpha}_n^i \wedge d\tilde{\alpha}_{-n}^i \right) \tag{4.3.34}$$

Now note that

$$-\frac{1}{n} d\alpha_{-n} \wedge d\alpha_n = \frac{1}{n} d\alpha_n \wedge d\alpha_{-n}$$

The above argument is true for $\tilde{\alpha}$ terms. We can thus consider sum over only positive values of n therefore introducing a term of 2 in equation 4.3.34:

$$\omega = i \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} \frac{1}{n} \left(d\alpha_n^i \wedge d\alpha_{-n}^i + d\tilde{\alpha}_n^i \wedge d\tilde{\alpha}_{-n}^i \right) \tag{4.3.35}$$

The above expression resembles the diagonal elements of a matrix. This matrix can then be written as:

$$\omega_{ij} = \frac{i}{n} \mathbb{1}_{d-2} \tag{4.3.36}$$

The inverse of the matrix then becomes:

$$\omega^{ij} = -in \mathbb{1}_{d-2} \tag{4.3.37}$$

Note that $\mathbb{1}_{d-2}$ represents the identity matrix of dimensions $d-2$.

The Poisson bracket relations then become:

$$\{\alpha_n^i, \alpha_m^j\} = \{\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\} = -in\, \delta^{ij} \delta_{n+m,0} \tag{4.3.38}$$

The commutation relations therefore become (for $\hbar = 1$):

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\, \delta^{ij} \delta_{n+m,0} \tag{4.3.39}$$

The above commutation relation is similar to that of harmonic oscillator modes. We define the creation and annihilation operators as $a_n = \frac{\alpha_n}{\sqrt{n}}$ and $a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{n}}$ respectively. We can then guess the Hamiltonian of the system analogous to that of harmonic oscillator;

$$H = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} (\alpha_n^i \alpha_{-n}^i + \alpha_{-n}^i \alpha_n^i) + \frac{1}{2} i \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} (\tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i) \tag{4.3.40}$$

The Hamiltonian of the system then seems to represent two infinite towers of creation and annihilation operators. Once we quantise the Hamiltonian, the consecutive energy levels would be of equal spacing of n. Each oscillator state would correspond to different particles in the system. Each of these particles must obey its own Schrodinger equation. The Hamiltonian then would help us to evaluate wavefunction of inifinite number of particles of ever increasing mass.

4.4 Quantising the String: Part II

In this section we carry out the process of quantising the string. We will emphasise on the case of zero modes, for we evaluated the Hamiltonian for non-zero modes in the previous section. Finally, we would also be reviewing one of the most puzzling results of analysing string theory.

4.4.1 The Level-Matching condition

Considering equation (4.3.30) we can evaluate:

$$\partial_+ X^i = \alpha' \frac{\partial}{\partial \sigma_+} \left(\frac{\sigma^+ + \sigma_-}{2} \right) + \frac{\partial}{\partial \sigma_+} \left[i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^i}{n} e^{-in\sigma_+} \right) \right]$$

$$\partial_+ X^i = \frac{\alpha' p^i}{2} - \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^i e^{-in\sigma_+} \tag{4.4.1}$$

Considering the fact that the cross-terms make the integral vanish, we can write;

$$\partial_+ X^i \partial_+ X_i = \left(\frac{\alpha' p^i}{2} \right)^2 + \frac{\alpha'}{2} \sum_{n \neq 0} \sum_{m \neq 0} \alpha_n^i \alpha_m^i e^{-i(m+n)\sigma_+} \tag{4.4.2}$$

From the discussion given in previous section we know that the integral remains only if $m = -n$. Now using the constraint equation (4.3.13) we get:

$$\left(\frac{\alpha' p^i}{2} \right)^2 + \frac{\alpha'}{2} \sum_{n \neq 0} \alpha_n^i \alpha_{-n}^i = 0 \tag{4.4.3}$$

We can convert the sum over all integral values of n to a sum over positive integral values of n by adding an extra term to the above equation. Also, we take note of the fact that the equation fixes $d-2$ degrees of freedom. Hence we can rewrite equation (4.4.3) as:

$$\sum_{i=1}^{d-2}\left(\frac{\alpha' p^i}{2}\right)^2 + \frac{\alpha'}{2}\sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\alpha_n^i\alpha_{-n}^i + \alpha_{-n}^i\alpha_n^i\right) = 0 \tag{4.4.4}$$

In the above given steps we applied the constraint equation to X^+. Recall that X^+ is a purely zero mode. Also the contribution of products of zero and non-zero modes vanishes when summed over all modes. Hence we can easily covariantise equation (4.4.4) by promoting conjugate momentum to include temporal contribution, i.e.; $p^i \to p^\mu$. Hence we reshape equation (4.4.4) as:

$$\sum_{i=1}^{d-2}\left(\frac{\alpha'}{2}\right)^2 p^\mu p_\mu + \frac{\alpha'}{2}\sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\alpha_n^i\alpha_{-n}^i + \alpha_{-n}^i\alpha_n^i\right) = 0 \tag{4.4.5}$$

The quantum mechanical operator for momentum is given by, $p_\mu = -i\partial_\mu$, which when squared over becomes; $p^\mu p_\mu = -\partial^\mu\partial_\mu$. Using Klein-Gordon equation we can infer that $\partial^\mu\partial_\mu := m^2$, m being mass of the particle. The equation (4.4.5) then translates to:

$$m^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\alpha_n^i\alpha_{-n}^i + \alpha_{-n}^i\alpha_n^i\right) \tag{4.4.6}$$

Using similar steps we can evaluate the equation given below:

$$m^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\tilde{\alpha}_n^i\tilde{\alpha}_{-n}^i + \tilde{\alpha}_{-n}\tilde{\alpha}_n\right) \tag{4.4.7}$$

The equations (4.4.6) and (4.4.7) reassert our claim that the Hamiltonian resembles to that of a harmonic oscillator, thus generating particles of different mass m as we vary value of n. Note that both equations give same mass value m if we have same value of n. Hence we can write:

$$\sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\alpha_n^i\alpha_{-n}^i + \alpha_{-n}^i\alpha_n^i\right) = \sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\left(\tilde{\alpha}_n^i\tilde{\alpha}_{-n}^i + \tilde{\alpha}_{-n}\tilde{\alpha}_n\right) \tag{4.4.8}$$

This is known as level-matching condition. It compares the levels of both right moving oscillators and left moving oscillators generated by α_n and $\tilde{\alpha_n}$ respectively.
We can normally order the level-matching condition by considering the commutation relation:

$$[\alpha_n^i, \alpha_{-n}^i] = n \tag{4.4.9}$$

We can easily derive, $\alpha_n^i \alpha_{-n}^i = n + \alpha_{-n}^i \alpha_n^i$ using the above commutation relation. Hence we can rewrite equation (4.4.6) as:

$$m^2 = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} \left(\frac{n}{2} + \alpha_{-n}^i \alpha_n^i \right) \tag{4.4.10}$$

Similarly we can rewrite equation (4.4.6) as:

$$m^2 = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} \left(\frac{n}{2} + \tilde{\alpha}_{-n} \tilde{\alpha}_n \right) \tag{4.4.11}$$

The level-matching condition can then be reframed as a normally ordered expression given by:

$$\sum_{n=1}^{\infty} \sum_{i=1}^{d-2} \alpha_{-n}^i \alpha_n^i = \sum_{n=1}^{\infty} \sum_{i=1}^{d-2} \tilde{\alpha}_{-n} \tilde{\alpha}_n \tag{4.4.12}$$

Note that $n/2$ term in equations (4.4.10) and (4.4.11) is divergent when n is summed till infinity. We will now attempt to account for this term.

4.4.2 Fixing the divergent term

Since the Hamiltonian appears to be analogous to that of harmonic oscillator, $n/2$ term seems to be contribution of zero point energies of the system. However the term is divergent and increases to infinity as n increases. This presents us with an ambiguity since we are not dealing with standard quantum field theory. Rather we are dealing with a theory coupled to 2d gravity as is clear by the choice of metric. Such gravity theories experience absolute energies and can't have infinite zero point energy. We must attempt to remove this ambiguity.

Let us add a counter term in the action so that it helps us to provide a physical meaning to the ambiguity. We therefore introduce cosmological constant, Λ to the action as follows:

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + A\Lambda^2\right) \tag{4.4.13}$$

where A is some number. Now imagine that the string is not periodic in 2π but has a length L. Then the term $A\Lambda^2$ instead becomes $A\Lambda^2 L$. We now attempt to find any term proportional to the string length so that it counters the cosmological cosntant-term. Any other terms (not proportional to length) must be interpreted as physical ones. Introducing the string length makes it necessary to fix the following term; $\sum_{n=1}^{\infty}\frac{n}{L}$.
To deal with divergent term, we wish to regulate the sum. We introduce the regulator as a function of physical momenta. If x is a variable depending on momenta, for small x the function must remain unity, while it must decay as x increase further. We choose $x = p/\Lambda$, hence for $p << \Lambda$, the summation remains unchanged. Assume that the physical momentum of the string is accounted by $p = n/\Lambda$. Hence the the function must be $f(n/L\Lambda)$. The regulated sum then becomes:

$$\sum_{n=1}^{\infty}\frac{n}{L}f\left(\frac{n}{L\Lambda}\right) \tag{4.4.14}$$

Using the result of Euler-Maclaurin formula we can evaluate the sum as:

$$\sum_{n=1}^{\infty}\frac{n}{L}f\left(\frac{n}{L\Lambda}\right) = \int_0^{\infty}\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)dn - \frac{1}{12}\left[\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)\right]'(0) + \frac{1}{4!30}\left[\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)\right]'''(0) + ... \tag{4.4.15}$$

We now analyse each term of the above expression.

- For first term we make the substitution $y = n/L\Lambda$, the integral then becomes:
$$\int_0^{\infty}\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)dn = L\Lambda^2\int_0^{\infty}yf(y)dy$$
The integral on the RHS just gives us a number say A. Therefore the first term gives a contribution of $AL\Lambda^2$ to the regulated sum.

- The second term contains a single derivative, again this derivative acts on n. The second term then gives a contribution:

$$\frac{1}{12}\left[\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)\right]'(0) = \frac{1}{12L}f(0) = \frac{1}{12L}$$

- The third term can be approximated as:

$$\frac{1}{4!30}\left[\frac{n}{L}f\left(\frac{n}{L\Lambda}\right)\right]'''(0) = \frac{1}{4!30}\frac{1}{L}\left(\frac{1}{L\Lambda}\right)^2 f''\left(\frac{0}{L\Lambda}\right)$$

  Since we take Λ is very large we can approximate the above equation to zero. Hence the third term and higher order terms can be approximated to zero.

The regulated sum in () then finally becomes:

$$\sum_{n=1}^{\infty}\frac{n}{L}f\left(\frac{n}{L\Lambda}\right) = AL\Lambda^2 - \frac{1}{12L} \tag{4.4.16}$$

The term $AL\Lambda^2$ can be negated by that in the action of string. However the term $1/12L$ seems to be a physical one. We must therefore attempt to account for this term using the physical notions of our theory.
Finally we write the regulated sum as:

$$\sum_{n=1}^{\infty}\frac{n}{L} = -\frac{1}{12L} \tag{4.4.17}$$

4.4.3 Proof of $d = 26$

Using the level matching condition we define the levels as:

$$N = \sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\alpha_{-n}^i\alpha_n^i \qquad \tilde{N} = \sum_{n=1}^{\infty}\sum_{i=1}^{d-2}\tilde{\alpha}_{-n}\tilde{\alpha}_n \tag{4.4.18}$$

Using these newly defined levels and equation (4.4.17) we can rewrite equations (4.4.10) and ((4.4.11)) as:

$$m^2 = \frac{4}{\alpha'}\left[N - \frac{d-2}{24}\right] \tag{4.4.19}$$

and,

$$m^2 = \frac{4}{\alpha'}\left[\tilde{N} - \frac{d-2}{24}\right] \tag{4.4.20}$$

It is important to note that the the theory we are constructing shall have a Hilbert space as a tensor product of an infinite number of wavefunctions of Harmonic oscillators tensored with a spatial wavefunction of d variables. Hence the ground state of such Hilbert space can be given by $|0\rangle \otimes \psi(x)$. In momentum space the ground state becomes, $|0; p\rangle$.
We wish to check the symmetries exhibited by the theory by considering different states in the Hilbert space.

The ground state: Tachyons

For ground state it is quite clear that the levels correspond to zero i.e., $N = \tilde{N} = 0$. Hence equations (4.4.19) and (4.4.20) become:

$$m^2 = -\frac{d-2}{6\alpha'} \tag{4.4.21}$$

These states therefore seem to resemble particles with negative mass-squared. Such particles are known as Tachyons and are claimed to travel faster than the speed of light! However we can get over this absurd interpretation by considering the field heuristic approach. We assume some field propagating in spacetime whose quanta are Tachyons. If $T(x)$ represents Tachyon field and $V(T)$ the potential then we can define:

$$m^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0}$$

The negative mass then conveys the idea that the potential is maximum at $T = 0$. This can be represented as shown in Figure 4.3.

A Digression: Wigner's Classification

Before moving forward to explore various symmetries obeyed by higher excited states we mention an important condition which helps us to do so. The Wigner's classification categorises the d dimensional theory for non-negative energies(masses) into two classes of irreducible and unitary representations of Poincare group.

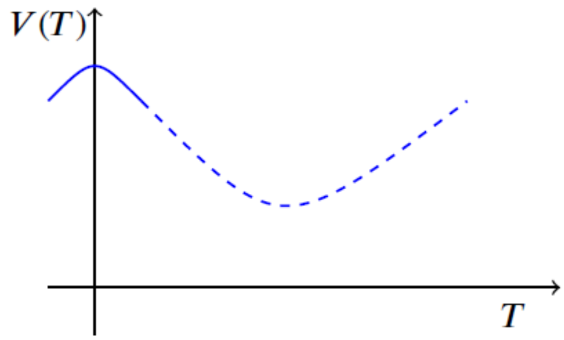

Figure 4.3: Representation of mass using Tachyon field

- The massless particles shall form $SO(d-2)$ group representation.
- The massive particles shall form $SO(d-1)$ group representation.

First Excited state: Massless

The first excited state can be given as: $\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle$. The state can thus be formed in $(d-2)^2$ ways, representing each $(d-2)$ contribution from $\tilde{\alpha}^i_{-1}$ and α^i_{-1}. These $(d-2)^2$ can never be packaged in $(d-1)$ representations. However we do have another condition namely that of the massless particles. Hence for the first excited state the particles must be massless i.e. $m=0$. Hence for $N=\tilde{N}=1$ we have $m=0$, the equations (4.4.19) and (4.4.20) then become:

$$0 = 1 - \frac{d-2}{24}$$

Hence we obtain:

$$d = 26 \tag{4.4.22}$$

The first excited state then corresponds to that of massless particles belonging to $SO(24)$ representation and quantising in 26 dimensions.

Higher excited states: Massive

It can be shown that the higher excited states correspond to that of massive particles provided that $d=26$. The states are represented as in $SO(25)$
This is a very important result implying that the whole theory is actually a

Lorrentz invariant theory provided that we set $d = 26$. In other words it can be stated that the Bosonic string theory quantises in 26 dimensions. The proof serves as a glimpse that string theory is a higher dimensional theory.

4.5 Concluding Remarks

We now conclude the final model of this thesis by reflecting on some main features of our analysis of Bosonic strings.

- Nambu-Gotto action and Polyakov action serve as the action principles for Bosonic strings.
- Light-cone gauge coordinates can be used for quantising the action principles.
- The cannonical quantisation of Bosonic strings makes it clear that the theory resembles to the case of infinite quantum harmonic oscillators generating particles of different masses at each particular level.
- The ground state of the quantum theory of Bosonic strings seems to generate Tachyons.
- The first excited state corresponds to the case of massless particles belonging to $SO(24)$ representation.
- The higher excited states correpond to the case of massive particles with increasing mass as level increases. These states transform in $SO(25)$ representation.
- The Bosonic string theory must quantise in $d = 26$ dimensions.

Chapter 5

Conclusion

In the thesis we glanced at three different models, in an attempt to develop mathematical as well as physical techniques in order to understand different aspects of theory of everything.
The first model presents a brief review of a rather mathematical approach to develop conformal field theory. We end the anlaysis by reviewing the Virasoro algebra. The analysis can be further extended to derive and understand different aspects of AdS/CFT correspondence principle. The AdS/CFT duality serves as a transitioning bridge between quantum field theory and a quantum theory of gravity [13]. On the other hand, it plays a vital role for understanding the Holographic principle [21]. Also, it is quite clear that the correspondence principle to understand the dynamics of quantum black holes [14]. The mathematical tools reviewed in the analysis of CFT can be carried over to derive different aspects of Bosonic strings in a more elegant way than that presented in this thesis [17]. CFT has also played a very important role in past few years, to analyse certain critical condensed matter systems [10][6].

The second model deals with study of some important aspects of evaporating black holes. It is very clear that black holes do behave like ordinary thermal objects adhering to the rules of quantum mechanics. On the other hand, the information paradox seems to violate unitarity. However, we conclude that quantum field theory calculations can be used to develop the replica wormhole model[2]. This thesis provides one with a qualitative review of the implications of this wormhole model. The model promises to resolve the information model and hence preserves unitary evolution of black holes. Further, it can be inferred that the black hole interior plays an important role

while contributing to evolution of black holes and the fate of infalling information. This has opened up new branch of theoretical physics connecting black hole interior with quantum information theory [1]. Study of black holes therefore seem to one among the very few areas of physics which require the unification of quantum mechanics and gravity. Hence, it is clear that the mysteries of black holes when solved lead us closer to finding the quantum theory of gravity.
In our final model we present a very brief review of Bosonic string theory. This quantum theory of one dimensional objects presents one with an astonishing model of infinite harmonic oscillators representing particles of different masses at each level. Using mathematical techniques of CFT, we can extend the Bosonic string theory to the full model known as Fermionic strings. The dimensions get limited to 11. The Fermionic string theory can then be extended to incorporate the study of D-branes[18]. These are particle-like objects which when collected in a large amount can warp the spacetime around them in a significant manner. The D-brane model is therefore suitable for studying black holes with a string theory perspective. It can resolve the information paradox, thus reaffirming the fact that black holes are governed by the rules of quantum mechanics.

We finally conclude by noting the fact that all the three models are interconnected and provide one with mesmerising results when associated with each other. The mathematical techniques discussed in this thesis, though being rudimentary, will surely be useful for one to be introduced to a rigorous approach required for developing various aspects of the quest for a theory of everything.

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